Math 350 — Some Extra Review Problems for Midterm 2

1. We know that the Lagrange functions satisfy $L_k(x_j) = \delta_{jk}$. This obviously implies

$$\sum_{k=1}^{n} L_k(x) = 1$$

provided x is one of the interpolation nodes $x_1, x_2, \ldots, x_n$.

   (a) Let $n = 2$ and show that the above equation is true for *all* values of x.

   (b) Show that the equation is even true for arbitrary values of n (and all values of x).

2. (a) Determine whether the function

$$S(x) = \begin{cases} x, & 0 \le x \le 1 \\ -\frac{1}{2}(2-x)^2 + \frac{3}{2}, & 1 \le x \le 2 \\ \frac{3}{2}, & 2 \le x \le 3 \end{cases}$$

   is a quadratic spline on $[0, 3]$.

   (b) Is it a cubic spline?

   (c) Sketch the graph of S.

3. Since the bisection method improves the accuracy of its approximation to the root of the equation $f(x) = 0$ by a factor of two in each iteration we are assured that the error e_n after n steps satisfies

$$e_n \le \frac{b-a}{2^{n+1}},$$

where $[a, b]$ is the interval of interest.

   (a) Let $[a, b] = [1.5, 2]$ and use the error estimate above to determine how many steps of the bisection method are required to compute $\sqrt{3}$ with an error no more than 10^{-2}, i.e., when solving the equation $x^2 - 3 = 0$.

   (b) Perform the actual calculations.

4. (a) Let $f(x) = -x^3 - \cos x$ and $x_0 = -1$. Use Newton's method to compute x_2.

   (b) What happens if you start the iteration with $x_0 = 0$?

5. Recall that Newton's method for systems of nonlinear equations is given by

   Input $\boldsymbol{f}$, J, $\boldsymbol{x}^{(0)}$

   for $n = 0, 1, 2, \ldots$ do

   Solve $J(\boldsymbol{x}^{(n)})\boldsymbol{h} = -\boldsymbol{f}(\boldsymbol{x}^{(n)})$ for $\boldsymbol{h}$

   Update $\boldsymbol{x}^{(n+1)} = \boldsymbol{x}^{(n)} + \boldsymbol{h}$

   end

   Output $\boldsymbol{x}^{(n+1)}$

where the Jacobian is of the form

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_m} \end{bmatrix}.$$

Perform two iterations of Newton's method starting with $\boldsymbol{x}^{(0)} = [0,1]^T$ to get an approximate solution of the system

$$\begin{aligned} 4x^2 - y^2 &= 0 \\ 4xy^2 - x &= 1. \end{aligned}$$

6. Suppose you are fitting a function of the form $f(t) = x_1 t + x_2 e^t$ to the three data points $(1,2)$, $(2,3)$, $(3,5)$.
   (a) Set up the overdetermined linear system $Ax = b$ for the least squares problem needed to find the coefficients x_1 and x_2.
   (b) Set up the corresponding normal equations.
   (c) Compute the least squares solution by solving the normal equations by hand.
7. A traitor in the Middle Ages was stretched on a rack to lengths $L = 5$, 6 and 7 feet under applied forces of $F = 1$, 2 and 4 tons. Assuming Hooke's law in the form $L = a + bF$, find his normal length a by least squares.
8. Describe how to derive a numerical integration formula based on a cubic interpolation polynomial at 4 equally spaced points for $\int_a^b f(x)dx$.