MATH 350 — Midterm 2, April 7, 2011

Make sure to show all your work!

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1. For any non-negative integer p the function

$$(x)_+^p = \begin{cases} x^p & \text{if } x > 0, \\ 0 & \text{otherwise} \end{cases}$$

is called the *truncated power function.*
Show that $s(x) = (x-1)_+^3$ is a cubic spline and sketch its graph for x in the interval $[0, 3]$.

4 $s(x) = \begin{cases} S_1(x) = (x-1)^3, & x > 1 \\ S_2(x) = 0 & \text{otherwise} \end{cases}$ so the only breakpoint is at $x=1$

Cubic spline checklist:

2 (1) cubic pieces: both s_1 and s_2 are cubic polynomials

2 (2) continuity at $x=1$: $s_1(1) = (1-1)^3 = 0$, $s_2(1) = 0$ so ok

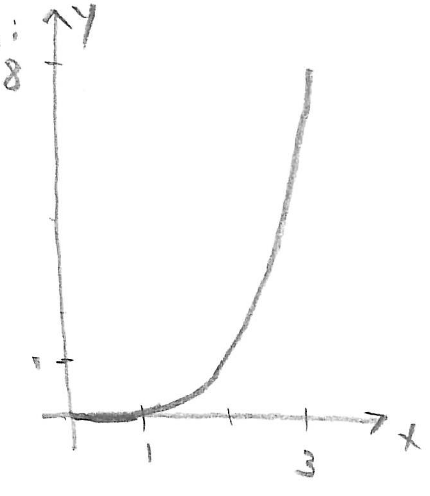
2 (3) continuity of s' at $x=1$: $s_1'(x) = 3(x-1)^2$, $s_2'(x) = 0$

so $s_1'(1) = 3(1-1)^2 = 0$, $s_2'(1) = 0$ ok

2 (4) continuity of s'' at $x=1$: $s_1''(x) = 6(x-1)$, $s_2''(x) = 0$

so $s_1''(1) = 6(1-1) = 0$, $s_2''(1) = 0$ ok

4 Sketch:



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2. Consider the nonlinear equation $x^2 e^x = 1$.

   (a) Show – using a rigorous argument – that this equation has a root in the interval $[0, 1]$.

   (b) Use the secant method with initial guesses $x_0 = 0$ and $x_1 = 1$ to obtain the approximate root x_4 (which should match the first significant digit of the "exact" solution $x = 0.7035$). Use at least 6 significant digits in your calculations.

(a) First convert to a zero finding problem, i.e., find x such that

$$f(x) = x^2 e^x - 1 = 0.$$

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A root in $[0,1]$ is guaranteed by the intermediate value theorem if

$$f(0) = 0^2 e^0 - 1 = -1 < 0$$

and $f(1) = 1^2 e^1 - 1 = e - 1 = 1.71828 > 0$

have different signs. $\to$ ok.

(b) The secant method is $x_{n+1} = x_n - \dfrac{f(x_n)}{\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}} = x_n - \dfrac{x_n - x_{n-1}}{\frac{f(x_{n-1})}{f(x_n)} - 1}$

3 Start with $x_0 = 0$, $x_1 = 1$. $n = 1, 2, 3, 4$

Then

3 $x_2 = 1 - \dfrac{1 - 0}{\frac{f(x_0)}{f(x_1)} - 1} \approx 0.367879$ and $f(x_2) = -0.804485$

3 $x_3 = 0.367879 - \dfrac{0.367879 - 1}{\frac{f(x_1)}{f(x_2)} - 1} \approx 0.569456$ and $f(x_3) = -0.426897$

3 $x_4 = 0.569456 - \dfrac{0.569456 - 0.367879}{\frac{f(x_2)}{f(x_3)} - 1} = \underline{\underline{0.797357}}$

[with $f(x_4) = 0.4112$ (still not that close to zero)]

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3. Consider the three U.S. population data points taken from `censusgui.m` (here x measures years since 1900 and y is the population rounded to the nearest million):

x	0	10	20
y	76	92	106

Since population growth is often assumed to follow an exponential curve, we want to fit the data with the model $y = ce^{\alpha x}$.

(a) Apply the natural logarithm to the exponential (i.e., nonlinear) model given above to convert it to a *linear* model for an appropriately transformed set of variables.

(b) Use the linear model you derived in (a) to obtain a prediction for the U.S. population in the year 1925 rounded to the nearest million.

(a) $y = ce^{\alpha x}$, so $\ln y = \ln(ce^{\alpha x}) = \ln c + \alpha x$

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Therefore, a linear model (for $z = \ln y$) is:

$$z = \alpha x + \beta, \quad \text{where } \beta = \ln c \text{ or } c = e^{\beta}$$

(b) The overdetermined linear system to solve is

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$$\begin{bmatrix} 0 & 1 \\ 10 & 1 \\ 20 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \ln 76 \\ \ln 92 \\ \ln 106 \end{bmatrix} = \begin{bmatrix} 4.3307 \\ 4.5218 \\ 4.6634 \end{bmatrix} \iff \underline{A}\underline{c} = \underline{z}$$

with normal equations $\underline{A}^T\underline{A}\,\underline{c} = \underline{A}^T\underline{z}$

$$\Rightarrow \begin{bmatrix} 500 & 30 \\ 30 & 3 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 10\ln 92 + 20\ln 106 \\ \ln 76 + \ln 92 + \ln 106 \end{bmatrix} = \begin{bmatrix} 138.4867 \\ 13.5160 \end{bmatrix}$$

$$\Rightarrow \quad \alpha = 0.0166 \quad \text{and} \quad \beta = 4.3390$$

Now $y(x) = ce^{\alpha x} = e^{\beta}e^{\alpha x} = e^{\alpha x + \beta}$

so that in 1925

$$y(25) = e^{(0.0166)(25) + 4.3390} = 116.1462 \approx 116 \text{ (million)}$$

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4. Consider the matrix

$$Q = \frac{1}{5}\begin{bmatrix} 0 & -4 & -3 \\ 4 & -\frac{9}{5} & \frac{12}{5} \\ \alpha & \beta & -\frac{16}{5} \end{bmatrix}.$$

Use the definition of an orthogonal matrix, $Q^TQ = I$, to find values of α and β that make Q orthogonal.

We need to find α and β such that $\underline{q_i}^T\underline{q_j} = \delta_{ij}$, where $\underline{q_i}$ is the j^{th} column of $\underline{Q}$.

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$$\underline{q_1}^T\underline{q_1} = \frac{1}{25}(0+16+\alpha^2) \overset{!}{=} 1 \Rightarrow \alpha = \pm 3 \quad (1)$$

$$\underline{q_1}^T\underline{q_2} = \frac{1}{25}\left(0 - \frac{36}{5} + \alpha\beta\right) \overset{!}{=} 0 \Rightarrow \alpha\beta = \frac{36}{5} \quad (2)$$

$$\underline{q_1}^T\underline{q_3} = \frac{1}{25}\left(0 + \frac{48}{5} - \frac{16\alpha}{5}\right) \overset{!}{=} 0 \Rightarrow 16\alpha = 48 \Rightarrow \underline{\underline{\alpha = 3}}$$

From (2) we then get $\underline{\underline{\beta}} = \frac{36}{5\alpha} = \underline{\underline{\frac{12}{5}}}$

Now we need to make sure that none of the other conditions

6 are violated if we use these values of α, β. ($\underline{q_1}^T\underline{q_1} = 1$ is ok from (1))

$$\underline{q_2}^T\underline{q_1} = \underline{q_1}^T\underline{q_2} = 0 \quad \text{so Ok}$$

$$\underline{q_2}^T\underline{q_2} = \frac{1}{25}\left(16 + \frac{81}{25} + \frac{144}{25}\right) = \frac{1}{25}\left(\frac{400+81+144}{25}\right) = \frac{625}{625} = 1 \quad \text{Ok}$$

$$\underline{q_2}^T\underline{q_3} = \underline{q_3}^T\underline{q_2} = \frac{1}{25}\left(12 - \frac{108}{25} - \frac{192}{25}\right) = \frac{1}{25}\left(\frac{300-108-192}{25}\right) = 0 \quad \text{Ok}$$

$$\underline{q_3}^T\underline{q_3} = \frac{1}{25}\left(9 + \frac{144}{25} + \frac{256}{25}\right) = \frac{1}{25}\left(\frac{225+144+256}{25}\right) = \frac{625}{625} = 1 \quad \text{Ok}$$

Note that the second part is necessary.
For example, if $\underline{q_2} = (1, -\frac{9}{5}, \frac{12}{5})^T$ then the first part would still be true, but $\underline{q_2}^T\underline{q_2} = \frac{2}{5} \neq 1$ (and also $\underline{q_2}^T\underline{q_3} \neq 0$).

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5. Consider the integral $I(f) = \int_0^1 \frac{1}{1+x^2}\,dx$ with "exact" value $I(f) = 0.78539816$.

(a) What is the expected accuracy of the composite trapezoidal rule T_n, i.e., how do you expect the approximation error $E_n(f) = |I(f) - T_n(f)|$ to change when you double the number of intervals used in your approximation from n to $2n$?

(b) Illustrate your statement made for part (a) by computing trapezoidal rule approximations T_n and associated errors E_n based on $n = 3$ and $n = 6$ subintervals.

(a) The composite trapezoidal rule has accuracy $O(h^2)$, where

6 $h = \frac{1}{n}$, so we should expect $\underline{E_{2n}(f) \approx \frac{1}{4} E_n(f)}$

(b) Take $a=0$, $b=1$, $f(x) = \frac{1}{1+x^2}$ and $h = \frac{b-a}{n} = \frac{1}{n}$

as well as $x_i = a + hi = \frac{i}{n}$, $i = 0, 1, \dots, n$,

in the trapezoidal rule

3 $$T_n(f) = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$$

Then

$$T_3(f) = \frac{1}{6}\left[f(0) + 2\left(f(\tfrac{1}{3}) + f(\tfrac{2}{3})\right) + f(1)\right]$$

3 $$= \frac{1}{6}\left[1 + 2\left(\frac{9}{10} + \frac{9}{13}\right) + \frac{1}{2}\right] \approx \underline{0.78076923}$$

with $E_3(f) = |I(f) - T_3(f)| = |0.78539816 - 0.78076923| = \underline{4.6289 \times 10^{-3}}$

and $$T_6(f) = \frac{1}{12}\left[f(0) + 2\left(f(\tfrac{1}{6}) + f(\tfrac{2}{6}) + f(\tfrac{3}{6}) + f(\tfrac{4}{6}) + f(\tfrac{5}{6})\right) + f(1)\right]$$

3 $$= \frac{1}{12}\left[1 + 2\left(\frac{36}{37} + \frac{36}{40} + \frac{36}{45} + \frac{36}{52} + \frac{36}{61}\right) + \frac{1}{2}\right] \approx \underline{0.78424077}$$

with $E_6(f) = |I(f) - T_6(f)| = |0.78539816 - 0.78424077| = \underline{1.1574 \times 10^{-3}}$

3 Note that $$\frac{E_6(f)}{E_3(f)} = \frac{1.1574 \times 10^{-3}}{4.6289 \times 10^{-3}} = 0.2500 = \underline{\frac{1}{4}}$$

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6. This problem looks at the use of the SVD to obtain the least squares solution of an over-determined linear system $Ac = y$. Consider the following MATLAB input and output:

```
>> A = [0.3111 1.1031; 1.5556 1.2728; 1.4708 1.3576];
>> y = [1 1 1]';
>> [U S V] = svd(A)
U =

    0.3333    0.9333    0.1332
    0.6667   -0.3333    0.6667
    0.6667   -0.1334   -0.7333

S =

    3.0000         0
         0    0.6000
         0         0

V =

    0.7071   -0.7071
    0.7071    0.7071
```

What output is produced when you next execute the following code?

```
r = length(find(diag(S)))
Uhat = U(:,1:r)
Shat = S(1:r,1:r)
z = Uhat'*y./diag(Shat)
c = V*z
```

$r = 2$

$$\text{Uhat} = \begin{bmatrix} 0.3333 & 0.9333 \\ 0.6667 & -0.3333 \\ 0.6667 & -0.1334 \end{bmatrix}$$

$$\text{Shat} = \begin{bmatrix} 3.0000 & 0 \\ 0 & 0.6000 \end{bmatrix}$$

Further,

$$\text{diag}(\text{Shat}) = \begin{bmatrix} 3.0000 \\ 0.6000 \end{bmatrix}$$

$$\text{Uhat}' * y = \begin{bmatrix} 0.3333 & 0.6667 & 0.6667 \\ 0.9333 & -0.3333 & -0.1334 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.6667 \\ 0.4667 \end{bmatrix}$$

$$z = \begin{bmatrix} \frac{1.6667}{3} & \frac{0.4667}{0.6} \end{bmatrix}' = \begin{bmatrix} 0.5556 \\ 0.7778 \end{bmatrix}$$

and

$$c = \begin{bmatrix} 0.7071 & -0.7071 \\ 0.7071 & 0.7071 \end{bmatrix} \begin{bmatrix} 0.5556 \\ 0.7778 \end{bmatrix} = \begin{bmatrix} -0.1571 \\ 0.9428 \end{bmatrix}$$