# MATH 461 — Test 1, October 6, 2015

1. Suppose a 1D rod has a constant internal heat source, so that the equation describing the heat flow within the rod is

$$\frac{\partial u}{\partial t}(x,t) = k\frac{\partial^2 u}{\partial x^2}(x,t) + 1, \qquad 0 < x < L.$$

Suppose we fix the boundary temperature to $u(0,t) = 0$ and $u(L,t) = 1$. What is the steady-state temperature of the rod?

For a steady-state problem we have $\frac{\partial u}{\partial t} \equiv 0$ so that we simply solve the ODE

$k\,u''(x) + 1 = 0$ or $u''(x) = -\frac{1}{k}$

with BCs $u(0) = 0$ and $u(L) = 1$.

Integration yields

$$u'(x) = -\frac{1}{k}x + c_1$$

$$u(x) = -\frac{1}{2k}x^2 + c_1 x + c_2$$

Use BCs to determine $c_1, c_2$:

$$u(0) = c_2 = 0$$

$$u(L) = -\frac{L^2}{2k} + c_1 L = 1 \quad \Rightarrow \quad c_1 = \frac{L}{2k} + \frac{1}{L}$$

and

$$\underline{\underline{u(x) = -\frac{1}{2k}x^2 + \left(\frac{L}{2k} + \frac{1}{L}\right)x}}$$

2. We know that the solution of the 1D periodic heat diffusion problem

$$\frac{\partial}{\partial t}u(x,t) = k\frac{\partial^2}{\partial x^2}u(x,t), \qquad -\pi < x < \pi,\ t > 0,$$
$$u(-\pi,t) = u(\pi,t), \quad \frac{\partial}{\partial x}u(-\pi,t) = \frac{\partial}{\partial x}u(\pi,t), \qquad t > 0,$$

is of the form

$$u(x,t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\,\mathrm{e}^{-n^2kt}.$$

Find the solution that satisfies the initial temperature distribution $u(x,0) = \sin^2(2x)$.

Hint: Recall the trigonometric identities $\sin^2 A = \frac{1}{2}(1 - \cos(2A))$ and $\cos^2 A = \frac{1}{2}(1 + \cos(2A))$.

We only need to consider

$$u(x,0) = \sin^2(2x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\underbrace{e^0}_{=1}$$

Therefore we need to find the Fourier coefficients of $f(x) = \sin^2(2x)$

Short answer: $f(x) = \frac{1}{2} - \frac{1}{2}\cos 4x$, so $a_0 = \frac{1}{2}$, $a_4 = -\frac{1}{2}$ and all other coefficients are zero.

Long answer:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^2 2x\, dx = \frac{1}{4\pi}\int_{-\pi}^{\pi}(1-\cos 4x)dx = \frac{1}{4\pi}\left[x - \frac{\sin 4x}{4}\right]_{-\pi}^{\pi} = \frac{2\pi}{4\pi} = \frac{1}{2}$$

$n>0$:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2 2x \cos nx\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}(1-\cos 4x)\cos nx\, dx$$

$$= \begin{cases} 0 & \text{if } n \neq 4 \text{ (by orthogonality of cosines)} \\ -\pi/2\pi = -\frac{1}{2} & \text{if } n = 4 \text{ (from orthogonality, or by integration)} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2 2x \sin nx\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}(1-\cos 4x)\sin nx\, dx = 0$$

by orthogonality

3. Consider Laplace's equation $\nabla^2 u(x,y) = 0$ on the rectangle $0 \le x \le L$, $0 \le y \le H$ with boundary conditions

$$u(0,y) = 0, \quad \frac{\partial}{\partial x} u(L,y) = \alpha u(L,y), \quad u(x,0) = 0, \quad u(x,H) = f(x), \qquad \alpha \ge 0.$$

**Do not solve this equation!** Apply only an appropriate separation of variables *Ansatz* and show that the positive eigenvalues must satisfy $\sqrt{\lambda} \cos \sqrt{\lambda} L - \alpha \sin \sqrt{\lambda} L = 0$. Do not worry about negative or zero eigenvalues.

Use the separation Ansatz $u(x,y) = \varphi(x) h(y)$.

Then $\nabla^2 u(x,y) = 0 \iff \varphi''(x) h(y) + \varphi(x) h''(y) = 0$

$$\iff \frac{\varphi''(x)}{\varphi(x)} = -\frac{h''(y)}{h(y)} = -\lambda$$

The eigenvalue problem is

$$\varphi''(x) = -\lambda \varphi(x) \quad \text{with BCs} \quad u(0,y) = 0 \Rightarrow \varphi(0) = 0$$

$$\frac{\partial u}{\partial x}(L,y) = \alpha u(L,y) \Rightarrow \varphi'(L) = \alpha \varphi(L)$$

For $\lambda > 0$ we have

$$\varphi(x) = c_1 \cos\sqrt{\lambda} x + c_2 \sin\sqrt{\lambda} x$$

and $\varphi(0) = c_1 = 0$

Also, $\varphi'(x) = c_2 \sqrt{\lambda} \cos\sqrt{\lambda} x$

So that

$$\varphi'(L) = \alpha \varphi(L) \iff c_2 \sqrt{\lambda} \cos\sqrt{\lambda} L = \alpha c_2 \sin\sqrt{\lambda} L$$

$$\overset{c_2 \neq 0}{\iff} \sqrt{\lambda} \cos\sqrt{\lambda} L - \alpha \sin\sqrt{\lambda} L = 0$$

4. (a) Show that the first three *Chebyshev polynomials of the first kind*, $T_0(x) = 1$, $T_1(x) = x$ and $T_2(x) = 2x^2 - 1$, are mutually orthogonal on the interval $[-1, 1]$ with respect to the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$. In other words, show that

$$\int_{-1}^{1} T_m(x)T_n(x)\frac{1}{\sqrt{1-x^2}}dx = 0 \text{ for } m, n = 0, 1, 2 \ m \neq n.$$

Hint: Since one can also write $T_n(x) = \cos(n \arccos x)$, the substitution $x = \cos\theta$ and the trigonometric identity $\cos^2 A = \frac{1}{2}(1 + \cos(2A))$ should help.

(b) Assuming that the orthogonality relations hold for all $m, n \geq 0$, how would you compute the *Fourier–Chebyshev coefficients* of a function $f$ that you want to represent by a *Fourier–Chebyshev series*

$$f(x) = \sum_{n=0}^{\infty} c_n T_n(x)?$$

You can obtain the formula for $c_n$ by following the same procedure we used to obtain the formulas for classical Fourier (sine and cosine) coefficients.

(a) Compute three integrals

$T_0 \perp T_1$:

$$\int_{-1}^{1} \underbrace{1}_{T_0(x)} \underbrace{x}_{T_1(x)} \underbrace{\frac{1}{\sqrt{1-x^2}}}_{w(x)} dx = \begin{bmatrix} x = \cos\theta \\ dx = -\sin\theta\, d\theta \end{bmatrix} = \int_{-\pi}^{0} \frac{\cos\theta}{\sin\theta}(-\sin\theta)\, d\theta$$

$$= \int_0^{\pi} \cos\theta \, d\theta = \sin\theta \Big|_0^{\pi} = 0$$

$T_0 \perp T_2$:

$$\int_{-1}^{1} 1(2x^2-1)\frac{1}{\sqrt{1-x^2}}\, dx = \begin{bmatrix} x = \cos\theta \\ dx = -\sin\theta\, d\theta \end{bmatrix} = \int_{\pi}^{0} \underbrace{\cos 2\theta}_{\text{trig. id.}} \frac{-\sin\theta}{\sin\theta}\, d\theta$$

$$= \int_0^{\pi} \cos 2\theta \, d\theta = \frac{\sin 2\theta}{2}\Big|_0^{\pi} = 0$$

$\underline{T_1 \perp T_2}$:

$$\int_{-1}^{1} x\,(2x^2-1)\frac{1}{\sqrt{1-x^2}}\,dx = \begin{bmatrix} x=\cos\theta \\ dx=-\sin\theta\,d\theta \end{bmatrix}$$

$$=\int_{\pi}^{0} \cos\theta\,(2\cos^2\theta-1)\;\frac{-\sin\theta}{\sin\theta}\,d\theta$$

$$= \int_0^{\pi}(2\cos^3\theta-\cos\theta)\,d\theta = 2\int_0^{\pi}\cos^3\theta\,d\theta - \underbrace{\int_0^{\pi}\cos\theta\,d\theta}_{=0 \text{ (see above)}}$$

$$=2\int_0^{\pi}\cos\theta\,(1-\sin^2\theta)\,d\theta = \begin{bmatrix} u=\sin\theta \\ du=\cos\theta\,d\theta \end{bmatrix}$$

$$=2\int_0^0 (1-u^2)\,du = 0$$

(b) Multiply both sides by $T_m(x)\frac{1}{\sqrt{1-x^2}}$ and integrate from $-1$ to $1$:

$$\int_{-1}^{1} f(x)\,T_m(x)\frac{1}{\sqrt{1-x^2}}\,dx = \sum_{n=0}^{\infty} c_n \underbrace{\int_{-1}^{1} T_n(x)\,T_m(x)\frac{1}{\sqrt{1-x^2}}\,dx}_{=0 \text{ if } m\neq n}$$

So

$$c_n = \frac{\int_{-1}^{1} f(x)\,T_n(x)\frac{1}{\sqrt{1-x^2}}\,dx}{\underbrace{\int_{-1}^{1} T_n^2(x)\frac{1}{\sqrt{1-x^2}}\,dx}_{=1 \text{ (but not part of this problem)}}}$$