3 Projectors

If $P \in \mathbb{C}^{m\times m}$ is a square matrix such that $P^2 = P$ then P is called a *projector*. A matrix satisfying this property is also known as an *idempotent* matrix.

Remark It should be emphasized that P need not be an orthogonal projection matrix. Moreover, P is usually not an orthogonal matrix.

Example Consider the matrix

$$P = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix},$$

where $c = \cos\theta$ and $s = \sin\theta$. This matrix projects perpendicularly onto the line with inclination angle θ in $\mathbb{R}^2$.

We can check that P is indeed a projector:

$$\begin{aligned} P^2 &= \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}\begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix} \\ &= \begin{bmatrix} c^4 + c^2s^2 & c^3s + cs^3 \\ c^3s + cs^3 & c^2s^2 + s^4 \end{bmatrix} \\ &= \begin{bmatrix} c^2(c^2+s^2) & cs(c^2+s^2) \\ cs(c^2+s^2) & s^2(c^2+s^2) \end{bmatrix} = P. \end{aligned}$$

Note that P is not an orthogonal matrix, i.e., $P^*P = P^2 = P \neq I$. In fact, $\operatorname{rank}(P) = 1$ since points on the line are projected onto themselves.

Example The matrix

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

is clearly a projector. Since the range of P is given by all points on the x-axis, and any point (x, y) is projected to $(x + y, 0)$, this is clearly not an orthogonal projection.

In general, for any projector P, any $v \in \operatorname{range}(P)$ is projected onto itself, i.e., $\boldsymbol{v} = P\boldsymbol{x}$ for some $\boldsymbol{x}$ then

$$P\boldsymbol{v} = P(P\boldsymbol{x}) = P^2\boldsymbol{x} = P\boldsymbol{x} = \boldsymbol{v}.$$

We also have

$$P(P\boldsymbol{v} - \boldsymbol{v}) = P^2\boldsymbol{v} - P\boldsymbol{v} = P\boldsymbol{v} - P\boldsymbol{v} = \boldsymbol{0},$$

so that $P\boldsymbol{v} - \boldsymbol{v} \in \operatorname{null}(P)$.

3.1 Complementary Projectors

In fact, $I - P$ is known as the *complementary projector* to P. It is indeed a projector since

$$(I-P)^2 = (I-P)(I-P) = I - \underbrace{IP}_{=P} - \underbrace{PI}_{=P} + \underbrace{P^2}_{=P} = I - P.$$

Lemma 3.1 *If P is a projector then*

$$\text{range}(I-P) = \text{null}(P), \tag{10}$$

$$\text{null}(I-P) = \text{range}(P). \tag{11}$$

Proof We show (10), then (11) will follow by applying the same arguments for $P = I - (I-P)$. Equality of two sets is shown by mutual inclusions, i.e., $A = B$ if $A \subseteq B$ and $B \subseteq A$.

First, we show $\text{null}(P) \subseteq \text{range}(I-P)$. Take a vector $\boldsymbol{v}$ such that $P\boldsymbol{v} = \boldsymbol{0}$. Then $(I-P)\boldsymbol{v} = \boldsymbol{v} - P\boldsymbol{v} = \boldsymbol{v}$. In words, any $\boldsymbol{v}$ in the nullspace of P is also in the range of $I-P$.

Now, we show $\text{range}(I-P) \subseteq \text{null}(P)$. We know that any $\boldsymbol{x} \in \text{range}(I-P)$ is characterized by

$$\boldsymbol{x} = (I-P)\boldsymbol{v} \quad \text{for some } \boldsymbol{v}.$$

Thus

$$\boldsymbol{x} = \boldsymbol{v} - P\boldsymbol{v} = -(P\boldsymbol{v} - \boldsymbol{v}) \in \text{null}(P)$$

since we showed earlier that $P(P\boldsymbol{v}-\boldsymbol{v}) = \boldsymbol{0}$. Thus if $\boldsymbol{x} \in \text{range}(I-P)$, then $\boldsymbol{x} \in \text{null}(P)$.
■

3.2 Decomposition of a Given Vector

Using a projector and its complementary projector we can decompose any vector $\boldsymbol{v}$ into

$$\boldsymbol{v} = P\boldsymbol{v} + (I-P)\boldsymbol{v},$$

where $P\boldsymbol{v} \in \text{range}(P)$ and $(I-P)\boldsymbol{v} \in \text{null}(P)$. This decomposition is *unique* since $\text{range}(P) \cap \text{null}(P) = \{\boldsymbol{0}\}$, i.e., the projectors are complementary.

3.3 Orthogonal Projectors

If $P \in \mathbb{C}^{m\times m}$ is a square matrix such that $P^2 = P$ and $P = P^*$ then P is called an *orthogonal projector*.

Remark In some books the definition of a projector already includes orthogonality. However, as before, P is in general *not* an orthogonal matrix, i.e., $P^*P = P^2 \neq I$.

3.4 Connection to Earlier Orthogonal Decomposition

Earlier we considered the orthonormal set $\{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n\}$, and established the decomposition

$$\begin{aligned} \boldsymbol{v} &= \boldsymbol{r} + \sum_{i=1}^{n} (\boldsymbol{q}_i^* \boldsymbol{v}) \boldsymbol{q}_i \\ &= \boldsymbol{r} + \sum_{i=1}^{n} (\boldsymbol{q}_i \boldsymbol{q}_i^*) \boldsymbol{v} \end{aligned} \tag{12}$$

with $\boldsymbol{r}$ orthogonal to $\{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n\}$. This corresponds to the decomposition

$$\boldsymbol{v} = (I - P)\boldsymbol{v} + P\boldsymbol{v}$$

with $P = \displaystyle\sum_{i=1}^{n} (\boldsymbol{q}_i \boldsymbol{q}_i^*)$.

Note that $\sum_{i=1}^{n} (\boldsymbol{q}_i \boldsymbol{q}_i^*) = QQ^*$ with $Q = [\boldsymbol{q}_1 \boldsymbol{q}_2 \ddots \boldsymbol{q}_n]$. Thus the orthogonal decomposition (12) can be rewritten as

$$\boldsymbol{v} = (I - QQ^*)\boldsymbol{v} + QQ^*\boldsymbol{v}. \tag{13}$$

It is easy to verify that QQ^* is indeed an orthogonal projection:

1. $(QQ^*)^2 = Q \underbrace{Q^*Q}_{=I} Q^* = QQ^*$ since Q has orthonormal columns (but not rows).

2. $(QQ^*)^* = QQ^*$.

Remark The orthogonal decomposition (13) will be important for the implementation of the QR decomposition later on. In particular we will use the rank-1 projector

$$P_{\boldsymbol{q}} = \boldsymbol{q}\boldsymbol{q}^*$$

which projects onto the direction $\boldsymbol{q}$ and its complement

$$P_{\perp \boldsymbol{q}} = I - \boldsymbol{q}\boldsymbol{q}^*.$$

Thus,

$$\boldsymbol{v} = (I - \boldsymbol{q}\boldsymbol{q}^*)\boldsymbol{v} + \boldsymbol{q}\boldsymbol{q}^*\boldsymbol{v},$$

or, more generally, orthogonal projections onto an arbitrary direction $\boldsymbol{a}$ is given by

$$\boldsymbol{v} = \left(I - \frac{\boldsymbol{a}\boldsymbol{a}^*}{\boldsymbol{a}^*\boldsymbol{a}}\right)\boldsymbol{v} + \frac{\boldsymbol{a}\boldsymbol{a}^*}{\boldsymbol{a}^*\boldsymbol{a}}\boldsymbol{v},$$

where we abbreviate $P_{\boldsymbol{a}} = \frac{\boldsymbol{a}\boldsymbol{a}^*}{\boldsymbol{a}^*\boldsymbol{a}}$ and $P_{\perp \boldsymbol{a}} = (I - \frac{\boldsymbol{a}\boldsymbol{a}^*}{\boldsymbol{a}^*\boldsymbol{a}})$.

As a further generalization we can consider orthogonal projection onto the range of a (full-rank) matrix A. Earlier, for the orthonormal basis $\{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n\}$ (the columns of Q) we had $P = QQ^*$. Now we require only that $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\}$ be linearly independent. In order to compute the projection P for this case we start with an arbitrary vector $\boldsymbol{v}$. We need to ensure that $P\boldsymbol{v} - \boldsymbol{v} \perp \text{range}(A)$, i.e., if $P\boldsymbol{v} \in \text{range}(A)$ then

$$\boldsymbol{a}_j^*(P\boldsymbol{v} - \boldsymbol{v}) = 0, \qquad j = 1, \ldots, n.$$

Now, since $P\boldsymbol{v} \in \text{range}(A)$ we know $P\boldsymbol{v} = A\boldsymbol{x}$ for some $\boldsymbol{x}$. Thus

$$\begin{aligned} \boldsymbol{a}_j^*(A\boldsymbol{x} - \boldsymbol{v}) = 0, \qquad j = 1, \ldots, n \\ A^*(A\boldsymbol{x} - \boldsymbol{v}) = 0 \end{aligned}$$

or

$$A^* A\boldsymbol{x} = A^*\boldsymbol{v}.$$

One can show that $(A^*A)^{-1}$ exists provided the columns of A are linearly independent (our assumption). Then

$$\boldsymbol{x} = (A^*A)^{-1}A^*\boldsymbol{v}.$$

Finally,

$$P\boldsymbol{v} = A\boldsymbol{x} = \underbrace{A(A^*A)^{-1}A^*}_{=P}\boldsymbol{v}.$$

Remark Note that this includes the earlier discussion when $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\}$ is orthonormal since then $A^*A = I$ and $P = AA^*$ as before.