Math 477 — Homework Assignment 6, due Nov.28, 2006

1. Show that the basic fixed-point iteration

$$M\boldsymbol{x}^{(k)} = N\boldsymbol{x}^{(k-1)} + \boldsymbol{b}$$

is equivalent to the following three steps:

Given $\boldsymbol{x}^{(k-1)}$

(i) compute the residual $\boldsymbol{r}^{(k-1)} = \boldsymbol{b} - A\boldsymbol{x}^{(k-1)}$,

(ii) solve $M\boldsymbol{z}^{(k-1)} = \boldsymbol{r}^{(k-1)}$ for $\boldsymbol{z}^{(k-1)}$,

(iii) define $\boldsymbol{x}^{(k)} = \boldsymbol{x}^{(k-1)} + \boldsymbol{z}^{(k-1)}$.

2. Using the notation of the previous problem, show that

$$\begin{aligned} \boldsymbol{r}^{(k)} &= NM^{-1}\boldsymbol{r}^{(k-1)} \\ \boldsymbol{z}^{(k)} &= M^{-1}N\boldsymbol{z}^{(k-1)}. \end{aligned}$$

3. Find the explicit form of the iteration matrix $G = M^{-1}N$ in the Gauss-Seidel method when

$$A = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}.$$

4. Give an example of a matrix A that is not diagonally dominant, yet the Gauss-Seidel method applied to $A\boldsymbol{x} = \boldsymbol{b}$ converges.