Math 478 — Homework Assignment 1, due Jan. 30, 2007

1. Show that the function $f(t,x) = x^2 e^{-t^2} \sin t$ is Lipschitz continuous for $x \in [0,2]$.
2. Find the Lagrange and Newton forms of the interpolating polynomial for the data

x	-2	0	1
$f(x)$	0	1	-1

   Write both polynomials in the form $a + bx + cx^2$ to verify that they are identical as functions.

3. The equation $x - 9^{-x} = 0$ has a solution in $[0,1]$. Find the interpolation polynomial on $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$ for the function on the left side of the equation. By setting the interpolation polynomial equal to 0 and solving the equation, find an approximate solution to the equation.
4. The polynomial p of degree $\leq n$ that interpolates a given functions f at $n+1$ prescribed nodes is uniquely defined. Hence, there is a mapping $f \mapsto p$. Denote this mapping by L and show that

$$Lf = \sum_{i=0}^{n} f(x_i)\ell_i.$$

   Show that L is linear, i.e., $L(af + bg) = aLf + bLg$, where f and g are given functions, and a, b are real constants.

5. (a) Approximate the function $f(x) = e^{x/2}$ over the interval $[1,9]$ by a fourth-degree polynomial in two ways: using a Taylor polynomial centered at $x_0 = 5$, and using the Lagrange form of the interpolating polynomial with $x_0 = 1$, $x_1 = 3$, $x_2 = 5$, $x_3 = 7$, and $x_4 = 9$.
   (b) Plot the error estimates for these two approaches (using Taylor's Theorem and the Lagrange form of the interpolating polynomial) for $x \in [0,12]$.
   (c) Use your favorite software to plot the actual error for these approximants on $[0,12]$. Comment.
6. The first U.S. postage stamp was issued in 1885, with the cost to mail a letter set at 2 cents. In 1917, the cost was raised to 3 cents but then was returned to 2 cents in 1919. In 1932, it was upped to 3 cents again, where it remained for 26 years. Then a series of increases took place as follows: 1958 = 4 cents, 1963 = 5 cents, 1968 = 6 cents, 1971 = 8 cents, 1974 = 10 cents, 1978 = 15 cents, 1981 = 18 cents in March and 20 cents in October, 1985 = 22 cents, 1988 = 25 cents, 1991 = 29 cents, 1995 = 32 cents, 1999 = 33 cents, 2001 = 34 cents, 2002 = 37 cents, and 2006 = 39 cents. Determine the Newton interpolation polynomial for these data. Based on this, when will it cost $1 to mail a letter? When will it cost $10, and when will it cost 42 cents?