change (or error) in $A$. Therefore, we must consider $A$ and the associated linear system to be ill conditioned.

**A Rule of Thumb.** If Gaussian elimination with partial pivoting is used to solve a well-scaled nonsingular system $Ax = b$ using $t$-digit floating-point arithmetic, then, assuming no other source of error exists, it can be argued that when $\kappa$ is of order $10^p$, the computed solution is expected to be accurate to at least $t - p$ significant digits, more or less. In other words, one expects to lose roughly $p$ significant figures. For example, if Gaussian elimination with 8-digit arithmetic is used to solve the $2 \times 2$ system given above, then only about $t - p = 8 - 6 = 2$ significant figures of accuracy should be expected. This doesn’t preclude the possibility of getting lucky and attaining a higher degree of accuracy—it just says that you shouldn’t bet the farm on it.

The complete story of conditioning has not yet been told. As pointed out earlier, it’s about three times more costly to compute $A^{-1}$ than to solve $Ax = b$, so it doesn’t make sense to compute $A^{-1}$ just to estimate the condition of $A$. Questions concerning condition estimation without explicitly computing an inverse still need to be addressed. Furthermore, liberties allowed by using the $\approx$ and $\leq$ symbols produce results that are intuitively correct but not rigorous. Rigor will eventually be attained—see Example 5.12.1 on p. 414.

**Exercises for section 3.8**

3.8.1. Suppose you are given that

$$A = \begin{pmatrix} 2 & 0 & -1 \\ -1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}$$

(a) Use the Sherman–Morrison formula to determine the inverse of the matrix $B$ that is obtained by changing the $(3, 2)$ entry in $A$ from 0 to 2.

(b) Let $C$ be the matrix that agrees with $A$ except that $c_{32} = 2$ and $c_{33} = 2$. Use the Sherman–Morrison formula to find $C^{-1}$.

3.8.2. Suppose $A$ and $B$ are nonsingular matrices in which $B$ is obtained from $A$ by replacing $A_{xj}$ with another column $b$. Use the Sherman–Morrison formula to derive the fact that

$$B^{-1} - A^{-1} = \frac{(A^{-1}b - e_j)[A^{-1}]_{jx}}{[A^{-1}]_{jx}b}.$$
3.8.3. Suppose the coefficient matrix of a nonsingular system $Ax = b$ is updated to produce another nonsingular system $(A + cd^T)z = b$, where $b, c, d \in \mathbb{R}^{n \times 1}$, and let $y$ be the solution of $Ay = c$. Show that $z = x - yd^T x / (1 + d^T y)$.

3.8.4. (a) Use the Sherman–Morrison formula to prove that if $A$ is nonsingular, then $A + \alpha c_i c_j^T$ is nonsingular for a sufficiently small $\alpha$.

(b) Use part (a) to prove that $I + E$ is nonsingular when all $e_{ij}$'s are sufficiently small in magnitude. This is an alternative to using the Neumann series argument.

3.8.5. For given matrices $A$ and $B$, where $A$ is nonsingular, explain why $A + \epsilon B$ is also nonsingular when the real number $\epsilon$ is constrained to a sufficiently small interval about the origin. In other words, prove that small perturbations of nonsingular matrices are also nonsingular.

3.8.6. Derive the Sherman–Morrison–Woodbury formula. Hint: Recall Exercise 3.7.11, and consider the product $\begin{pmatrix} 1 & C \\ 0 & I \end{pmatrix} \begin{pmatrix} A & C \\ D^T & -I \end{pmatrix} \begin{pmatrix} I & 0 \\ D & I \end{pmatrix}$.

3.8.7. Using the norm (3.8.7), rank the following matrices according to their degree of ill-conditioning:

$A = \begin{pmatrix} 100 & 0 & 100 \\ 0 & 100 & -100 \\ 100 & -100 & 300 \end{pmatrix}$, \quad $B = \begin{pmatrix} 1 & 8 & -1 \\ -9 & -71 & 11 \\ 1 & 17 & 18 \end{pmatrix}$.

$C = \begin{pmatrix} 1 & 2 \ 2 & -42 \\ 0 & 1 & -15 \\ -45 & -9 & 1 \end{pmatrix}$.

3.8.8. Suppose that the entries in $A(t)$, $x(t)$, and $b(t)$ are differentiable functions of a real variable $t$ such that $A(t)x(t) = b(t)$.

(a) Assuming that $A(t)^{-1}$ exists, explain why $\frac{dA(t)^{-1}}{dt} = A(t)^{-1}A'(t)A(t)^{-1}$.

(b) Derive the equation

$x'(t) = A(t)^{-1}b'(t) - A(t)^{-1}A'(t)x(t)$.

This shows that $A^{-1}$ magnifies both the change in $A$ and the change in $b$, and thus it confirms the observation derived from (3.8.8) saying that the sensitivity of a nonsingular system to small perturbations is directly related to the magnitude of the entries in $A^{-1}$. 
Transposition and Rank

Transposition does not change the rank—i.e., for all $m \times n$ matrices,

$$\text{rank} (A) = \text{rank} (A^T) \quad \text{and} \quad \text{rank} (A) = \text{rank} (A^*). \quad (3.9.11)$$

Proof. Let $\text{rank} (A) = r$, and let $P$ and $Q$ be nonsingular matrices such that

$$PAQ - N_r = \begin{pmatrix} I_r & 0_{r \times n-r} \\ 0_{m-r \times r} & 0_{m-r \times n-r} \end{pmatrix}.$$

Applying the reverse order law for transposition produces $Q^T A^T P^T = N_r^T$.

Since $Q^T$ and $P^T$ are nonsingular, it follows that $A^T \sim N_r^T$, and therefore

$$\text{rank} (A^T) = \text{rank} (N_r^T) = \text{rank} \begin{pmatrix} I_r & 0_{r \times m-r} \\ 0_{n-r \times r} & 0_{n-r \times m-r} \end{pmatrix} = r = \text{rank} (A).$$

To prove $\text{rank} (A) = \text{rank} (A^*)$, write $N_r = \overline{N}_r = \overline{PAQ} = \overline{P \overline{A} Q}$, and use the fact that the conjugate of a nonsingular matrix is again nonsingular (because $K^{-1} = K^{-T}$) to conclude that $N_r \sim \overline{A}$, and hence $\text{rank} (A) = \text{rank} (\overline{A})$. It now follows from $\text{rank} (A) = \text{rank} (A^T)$ that

$$\text{rank} (A^*) - \text{rank} (A^T) = \text{rank} (A) = \text{rank} (A). \quad \blacksquare$$

Exercises for section 3.9

3.9.1. Suppose that $A$ is an $m \times n$ matrix.

(a) If $[A|I_m]$ is row reduced to a matrix $[B|P]$, explain why $P$ must be a nonsingular matrix such that $PA = B$.

(b) If $[\overline{A}|I_n]$ is column reduced to $[C|Q]$, explain why $Q$ must be a nonsingular matrix such that $AQ = C$.

(c) Find a nonsingular matrix $P$ such that $PA - E_A$, where

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 7 \\ 1 & 2 & 3 & 6 \end{pmatrix}.$$

(d) Find nonsingular matrices $P$ and $Q$ such that $PAQ$ is in rank normal form.
3.9.2. Consider the two matrices
\[
A = \begin{pmatrix} 2 & 2 & 0 & -1 \\ 3 & -1 & 4 & 0 \\ 0 & -8 & 8 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & -6 & 8 & 2 \\ 5 & 1 & 4 & -1 \\ 3 & -9 & 12 & 3 \end{pmatrix}.
\]
(a) Are \( A \) and \( B \) equivalent?
(b) Are \( A \) and \( B \) row equivalent?
(c) Are \( A \) and \( B \) column equivalent?

3.9.3. If \( A \overset{\text{row}}{\sim} B \), explain why the basic columns in \( A \) occupy exactly the same positions as the basic columns in \( B \).

3.9.4. A product of elementary interchange matrices i.e., elementary matrices of Type I—is called a permutation matrix. If \( P \) is a permutation matrix, explain why \( P^{-1} = P^T \).

3.9.5. If \( A_{n \times n} \) is a nonsingular matrix, which (if any) of the following statements are true?
(a) \( A \sim A^{-1} \).
(b) \( A \overset{\text{row}}{\sim} A^{-1} \).
(c) \( A \overset{\text{col}}{\sim} A \).
(d) \( A \sim I \).
(e) \( A \overset{\text{row}}{\sim} I \).
(f) \( A \overset{\text{col}}{\sim} I \).

3.9.6. Which (if any) of the following statements are true?
(a) \( A \sim B \implies A^T \sim B^T \).
(b) \( A \overset{\text{row}}{\sim} B \implies A^T \overset{\text{row}}{\sim} B^T \).
(c) \( A \overset{\text{col}}{\sim} B \implies A^T \overset{\text{col}}{\sim} B^T \).
(d) \( A \overset{\text{row}}{\sim} B \implies A \sim B \).
(e) \( A \overset{\text{col}}{\sim} B \implies A \sim B \).
(f) \( A \sim B \implies A \overset{\text{row}}{\sim} B \).

3.9.7. Show that every elementary matrix of Type I can be written as a product of elementary matrices of Types II and III. \textbf{Hint:} Recall Exercise 1.2.12 on p. 14.

3.9.8. If \( \text{rank}(A_{m \times n}) = r \), show that there exist matrices \( B_{m \times r} \) and \( C_{r \times n} \) such that \( A = BC \), where \( \text{rank}(B) = \text{rank}(C) = r \). Such a factorization is called a full-rank factorization. \textbf{Hint:} Consider the basic columns of \( A \) and the nonzero rows of \( E_A \).

3.9.9. Prove that \( \text{rank}(A_{m \times n}) = 1 \) if and only if there are nonzero columns \( u_{m \times 1} \) and \( v_{n \times 1} \) such that
\[
A = uv^T.
\]

3.9.10. Prove that if \( \text{rank}(A_{m \times n}) = 1 \), then \( A^2 = rA \), where \( r = \text{trace}(A) \).