Math 577 — Homework Assignment 4, due Oct.26, 2006

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1. Given $A \in \mathbb{C}^{m\times n}$ of rank n and $\boldsymbol{b} \in \mathbb{C}^m$, consider the block 2×2 system of equations

$$\begin{bmatrix} I & A \\ A^* & O \end{bmatrix} \begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix},$$

where I is the $m\times m$ identity matrix. Show that this system has a unique solution $[\boldsymbol{r}, \boldsymbol{x}]^T$, and that the vectors $\boldsymbol{r}$ and $\boldsymbol{x}$ are the residual and the solution of the least squares problem:

> Given $A \in \mathbb{C}^{m\times n}$ of full rank, $m \geq n$, $\boldsymbol{b} \in \mathbb{C}^m$, find $\boldsymbol{x} \in \mathbb{C}^n$ such that $\|\boldsymbol{b} - A\boldsymbol{x}\|$ is minimized.

2. Suppose $A \in \mathbb{C}^{m\times m}$ so that its upper-left $k\times k$ blocks $A(1:k, 1:k)$ are nonsingular (so that existence of an LU factorization is guaranteed) and that A is *banded* with bandwidth $2p+1$, i.e., $a_{ij} = 0$ for $|i-j| > p$. What can you say about the sparsity pattern of the factors L and U of A?

3. Suppose an $m\times m$ matrix A is written in the block form $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where A_{11} is $n\times n$ and A_{22} is $(m-n)\times(m-n)$. Assume that A is such that its LU factorization exists. Verify the formula

$$\begin{bmatrix} I & O \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ O & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$$

for "elimination" of the block A_{21}. The matrix $A_{22} - A_{21}A_{11}^{-1}A_{12}$ is known as the *Schur complement* of A_{11} in A.

4. Let A be the 4×4 matrix

$$A = \begin{bmatrix} -1 & 1 & 0 & -3 \\ 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 3 & 0 & 1 & 2 \end{bmatrix}.$$

   (a) Compute the LU factorization of A with and without partial pivoting.

   (b) Determine $\det(A)$ from the 2 LU factorizations of A obtained in (a).

   (c) Describe how Gaussian elimination with partial pivoting can be used to find the determinant of a general square matrix.

5. Let A be a nonsingular square matrix and let $A = QR$ and $A^*A = U^*U$ be QR and Cholesky factorizations, respectively, with the usual normalizations $r_{jj}, u_{jj} > 0$. Is it true or false that $R = U$? Explain.

6. Fill in the details needed in class, i.e., show that $\boldsymbol{x}^*A\boldsymbol{y} = \overline{\boldsymbol{y}^*A\boldsymbol{x}}$ provided A is Hermitian.

7. Prove another claim made in class: Show $A \in \mathbb{C}^{m\times m}$ is positive definite and $X \in \mathbb{C}^{m\times m}$ has full rank if and only if X^*AX is positive definite.