Math 577: Midterm — Tuesday, Oct. 17, 2006

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1. TRUE or FALSE?

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T (a) An overdetermined linear least squares problem $A\boldsymbol{x} \approx \boldsymbol{b}$ always has a unique solution $\boldsymbol{x}$ that minimizes the Euclidean norm of the residual vector $\boldsymbol{r} = \boldsymbol{b} - A\boldsymbol{x}$.

F (b) A good algorithm will produce an accurate solution regardless of the condition of the problem being solved.

F (c) Fitting a straight line to a set of data points is a linear least squares problem, whereas fitting a quadratic polynomial to the data is a nonlinear least squares problem.

T (d) A problem is ill-conditioned if its solution is highly sensitive to small changes in the problem data.

2. If A and B are $m \times m$ matrices, with A nonsingular, and $\boldsymbol{c}$ is an m-vector, how would you efficiently compute the product $A^{-1}B\boldsymbol{c}$?

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Use LU factorization $A = LU$

and solve $Ax = Bc \iff x = A^{-1}Bc$

Thus

$x = A^{-1}Bc \iff LUx = Bc$

1) Solve $Ly = Bc$ for y (lower triang)

2) Solve $Ux = y$ for x (upper triang)

Then $x = A^{-1}Bc$

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3. Assume $A \in \mathbb{R}^{m\times n}$ with $\text{rank}(A) = n$. Show that $\|A(A^TA)^{-1}A^T\|_2 = 1$.

Use SVD $\quad A = U\Sigma V^*, \quad A^T = V\Sigma U^*$

So $A(A^TA)^{-1}A^T = U\Sigma V^T(V\Sigma \underbrace{U^TU}_{=I}\Sigma V^T)^{-1}V\Sigma U^T$

$$= U\Sigma V^T \underbrace{(V\Sigma^2 V^T)}^{-1} V\Sigma U^T$$

invertible, since full rank

$$= V\Sigma^{-2}V^T$$

$$= U\Sigma \underbrace{V^TV}_{=I}\Sigma^{-2}\underbrace{V^TV}_{=I}\Sigma U^T$$

$$= U\underbrace{\Sigma\Sigma^{-2}\Sigma}_{=I} U^T = UU^T$$

So $\|A(A^TA^{-1})A^T\|_2 = \|UU^T\|_2 = \|U^T\|_2 = \|I\|_2 = 1$

since U orthogonal

4. Let V be the subspace of $\mathbb{R}^3$ spanned by $v_1 = [1\ 1\ 1]^T$ and $v_2 = [1\ 0\ -1]^T$.

15 (a) Find the orthogonal projection onto V.

5 (b) Find the point in V that is closest to $z = [1\ 2\ 1]^T$ if measured in the 2-norm.

a) Form $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}$

and $P = A(A^TA)^{-1}A^T$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow (A^TA)^{-1} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix}$$

and
$$\underline{P} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} = \underline{\begin{bmatrix} 5/6 & 1/3 & -1/6 \\ 1/3 & 1/3 & 1/3 \\ -1/6 & 1/3 & 5/6 \end{bmatrix}}$$

b) Project

$$Pz = \begin{bmatrix} 5/6 & 1/3 & -1/6 \\ 1/3 & 1/3 & 1/3 \\ -1/6 & 1/3 & 5/6 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \underline{\begin{bmatrix} 4/3 \\ 4/3 \\ 4/3 \end{bmatrix}}$$

5. Two matrices $A, B \in \mathbb{R}^{m\times m}$ are called *orthogonally equivalent* if $A = QBQ^T$ for some orthogonal matrix Q.

15 (a) Show that if A, B are orthogonally equivalent then they have the same singular values.

5 (b) Is the converse also true?

(a) Use SVD for $B = U\Sigma V^T$

Then $A = QU\Sigma V^T Q^T$

$= \tilde{U} \Sigma \tilde{V}^T$ same sing. values as B

$\tilde{U}, \tilde{V}$ are orthogonal since

$$(QU)^T(QU) = U^T \underbrace{Q^T Q}_{=I} U = I$$

since Q, U orthogonal

Same for $\tilde{V} = (QV)$

(b) <u>Not</u> true

Take $A = U_1 \Sigma V_1^T$, $B = U_2 \Sigma V_2^T$.

$\Leftrightarrow \Sigma = U_2^T B V_2$

then $A = \underbrace{U_1 U_2^T}\, B\, \underbrace{V_2 V_1^T}$

in general different (not Q, Q^T)

6. Consider the 2×2 matrix

$$A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}.$$

10 (a) Determine an SVD $A = U\Sigma V^T$ of A.

4 (b) What are the 1-, 2-, and ∞-norms of A?

8 (c) Find A^{-1} using the SVD computed in (a).

4 (d) What is the area of the ellipse onto which A maps the unit ball of $\mathbb{R}^2$? (Hint: The area of an ellipse given in normal form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab.)

(a) $A^T A = \begin{bmatrix} 104 & -72 \\ -72 & 146 \end{bmatrix} \Rightarrow \begin{matrix} \lambda_1 = 200 \\ \lambda_2 = 50 \end{matrix} \Rightarrow \begin{matrix} \sigma_1 = 10\sqrt{2} \\ \sigma_2 = 5\sqrt{2} \end{matrix}$

$v_1 = \begin{bmatrix} 1 \\ -4/3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4/3 \\ 1 \end{bmatrix}$ normalized $\Rightarrow V = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ -4 & +3 \end{bmatrix}$

$u_1 = \frac{1}{\sigma_1} A v_1 = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}, \quad u_2 = \frac{1}{5\sqrt{2}} A v_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$

$$\Rightarrow A = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix} \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} \end{bmatrix} \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix} \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$$

(b) $\|A\|_1 = \text{max col. sum} = \underline{16}$

$\|A\|_2 = \sigma_1 = \underline{10\sqrt{2}}$

$\|A\|_\infty = \text{max row sum} = \underline{15}$

(c) $A^{-1} = V\Sigma^{-1}U^T = \frac{1}{10}\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{10\sqrt{2}} & 0 \\ 0 & \frac{1}{5\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} \end{bmatrix} = \underline{\begin{bmatrix} 1/20 & -11/100 \\ 1/10 & -1/50 \end{bmatrix}}$

(d) ~~area~~ $= \pi ab$ with $\left.\begin{matrix} a = \|\sigma_1 u_1\| = \sigma_1 \\ b = \|\sigma_2 u_2\| = \sigma_2 \end{matrix}\right\} \Rightarrow \text{area} = \underline{\pi\, 100}$