# Math 578 — Homework Assignment 1, due Jan. 30, 2007

1. Show that the function $f(t,x) = x^2 e^{-t^2} \sin t$ is Lipschitz continuous for $x \in [0,2]$.

2. Find the Lagrange and Newton forms of the interpolating polynomial for the data

| $x$ | $-2$ | $0$ | $1$ |
| --- | --- | --- | --- |
| $f(x)$ | $0$ | $1$ | $-1$ |

.

Write both polynomials in the form $a + bx + cx^2$ to verify that they are identical as functions.

3. The polynomial $p$ of degree $\le n$ that interpolates a given functions $f$ at $n+1$ prescribed nodes is uniquely defined. Hence, there is a mapping $f \mapsto p$. Denote this mapping by $L$ and show that

$$Lf = \sum_{i=0}^{n} f(x_i)\ell_i.$$

Show that $L$ is linear, i.e., $L(af + bg) = aLf + bLg$, where $f$ and $g$ are given functions, and $a, b$ are real constants.

4. Prove that the mapping, $L$, in Problem 3 has the property that $Lq = q$ for every polynomial $q$ of degree at most $n$.

5. Prove that if we take *any* set of 23 nodes in the interval $[-1,1]$ and interpolate the function $f(x) = \cosh x$ with a polynomial $p$ of degree 22, then the relative error $|p(x) - f(x)|/|f(x)|$ is no greater than $5 \times 10^{-16}$ on $[-1,1]$.

6. (a) Approximate the function $f(x) = e^{x/2}$ over the interval $[1,9]$ by a fourth-degree polynomial in two ways: using a Taylor polynomial centered at $x_0 = 5$, and using the Lagrange form of the interpolating polynomial with $x_0 = 1$, $x_1 = 3$, $x_2 = 5$, $x_3 = 7$, and $x_4 = 9$.

   (b) Plot the error estimates for these two approaches (using Taylor's Theorem and the Lagrange form of the interpolating polynomial) for $x \in [0,12]$.

   (c) Use your favorite software to plot the actual error for these approximants on $[0,12]$. Comment.

7. Prove that

$$\det \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{vmatrix} = \prod_{0 \le j < k \le n} (x_k - x_j)$$

$$= (x_n - x_0)(x_n - x_1)(x_n - x_2)\cdots(x_n - x_{n-1})\cdots (x_3 - x_0)(x_3 - x_1)(x_3 - x_2)(x_2 - x_0)(x_2 - x_1)(x_1 - x_0).$$