Math 578: Midterm — Tuesday, March 20, 2007

SHOW ALL WORK! USE THESE SHEETS ONLY.

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1. Write down the linear Lagrange interpolation polynomial for the function $f(x) = x^3$ using the points $x_0 = 0$ and $x_1 = b$. Verify formula (7) in Theorem 1.9 of the class notes by direct calculation. In particular, show that in this case ξ has the unique value $\xi = \frac{1}{3}(x+b)$.

Data:

x	0	b
y	0	b^3

So Lagrange polynomial is

$$\underline{\underline{p_1(x)}} = \sum_{j=0}^{1} l_j(x) f(x_j) = l_1(x)\, b^3 = \frac{x-x_0}{x_1-x_0} b^3 = \frac{x}{b} b^3 = \underline{\underline{b^2 x}}$$

Difference:

$$f(x) - p(x) = x^3 - b^2 x = x(x^2 - b^2) = x(x-b)(x+b) \quad (1)$$

Estimate from Thm 1.9:

$$f(x) - p(x) = \frac{1}{2!} f''(\xi)(x-x_0)(x-x_1)$$

$$= \frac{1}{2} 6\xi \, x(x-b) \qquad (\text{since } f''(x) = 6x)$$

$$= 3\xi \, x(x-b) \quad (2)$$

Comparing (1) and (2) we see

$$3\xi = x+b \quad \text{or} \quad \underline{\underline{\xi = \frac{1}{3}(x+b)}}$$

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2. (a) Find a formula of the form $\int_0^{2\pi} f(x)dx = A_1 f(0) + A_2 f(\pi)$ that is exact for any function having the form $f(x) = a\cos x + b$.

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(b) Prove that the formula derived in (a) is exact for any function of the form

$$f(x) = \sum_{k=0}^{n} [a_k \cos(2k+1)x + b_k \sin kx] .$$

(a) Since $1, \cos x$ are lin independent on $[0, 2\pi]$ we can use them as basis and check

$$\left.\begin{aligned} \int_0^{2\pi} dx &= 2\pi = A_1 + A_2 \\ \text{and} \quad \int_0^{2\pi} \cos x \, dx &= 0 = A_1 - A_2 \end{aligned}\right\} \Rightarrow \underline{\underline{A_1 = A_2 = \pi}}$$

(b) Again, $\{\cos(2k+1)x, \sin kx, \; k = 0, \dots, n\}$ are linearly independent on $[0, 2\pi]$

So check only if exact for them:

$$\int_0^{2\pi} \cos(2k+1)x \, dx = 0 \stackrel{?}{=} A_1 - A_2 = \pi - \pi = 0 \quad \checkmark$$

$$\int_0^{2\pi} \sin kx \, dx = 0 = 0A_1 + 0A_2 \quad \checkmark$$

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3. Is it possible to find coefficients a, b, c, and d such that the function

$$S(x) = \begin{cases} 1 - 2x, \quad =: S_0(x) & -\infty < x \le -3, \\ a + bx + cx^2 + dx^3, \quad =: S_1(x) & -3 \le x \le 4, \\ 157 - 32x, \quad =: S_2(x) & 4 \le x < \infty, \end{cases}$$

is a natural cubic spline for the interval $[-3, 4]$?

Check C^0 continuity:

$$S_0(-3) = 7 \overset{!}{=} S_1(-3) = a - 3b + 9c - 27d \qquad (1)$$

$$S_1(4) = a + 4b + 16c + 64d \overset{!}{=} S_2(4) = 29 \qquad (2)$$

C^1 continuity:

$$S_0'(-3) = -2 \overset{!}{=} S_1'(-3) = b + 2cx + 3dx^2 \Big|_{x=-3} = b - 6c + 27d \quad (3)$$

$$S_1'(4) = b + 2cx + 3dx^2 \Big|_{x=4} = b + 8c + 48d \overset{!}{=} S_2'(4) = -32 \quad (4)$$

C^2 continuity:

$$S_0''(-3) = 0 \overset{!}{=} S_1''(-3) = 2c + 6dx \Big|_{x=-3} = 2c - 18d \qquad (5)$$

$$S_1''(4) = 2c + 6dx \Big|_{x=4} = 2c + 24d \overset{!}{=} 0 = S_2''(4) \qquad (6)$$

From (5) and (6) we have $\underline{\underline{c = d = 0}}$

but then $(3) \Rightarrow -2 = b$
and $(4) \Rightarrow -32 = b$ } a <u>contradiction</u>

<u>not possible</u>

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4. (a) Apply Richardson extrapolation to Euler's method using step sizes h and $h/2$ to derive the second-order Runge-Kutta method (modified Euler method) $y_{n+1} = y_n + hk_2$, where

$$k_1 = f(t_n, y_n), \quad k_2 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1).$$

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(b) How can this method be used to produce a table of values for the function $f(x) = \int_0^x e^{-t^2} dt$ at 100 equally spaced points on $[0, 1]$?

(a) Euler $y_{n+1} = y_n + hf(t_n, y_n)$ is $O(h)$

Richardson if $F = F_h + O(h^p)$, then

$$F \approx \frac{2^p}{2^p-1}\left[F_{h/2} - \frac{F_h}{2^p}\right] \quad p=1 \quad = 2F_{h/2} - F_h$$

F_h: $y_{n+1} = y_n + h f(t_n, y_n)$

$F_{h/2}$: $\tilde{y}_{n+1} = y_n + \frac{h}{2} f(t_n, y_n)$

and $y_{n+1} = \tilde{y}_{n+1} + \frac{h}{2} f(t_n + \frac{h}{2}, \tilde{y}_{n+1}) = y_n + \frac{h}{2} f(t_n, y_n) + \frac{h}{2} f(t_n + \frac{h}{2}, y_n + \frac{h}{2} f(t_n,$

$$= y_n + \frac{h}{2}\Big[\underbrace{f(t_n, y_n)}_{=k_1} + \underbrace{f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1)}_{=k_2}\Big]$$

$$= y_n + \frac{h}{2}[k_1 + k_2]$$

Now combine

$$2F_{h/2} - F_h = 2\left[y_n + \frac{h}{2}(k_1 + k_2)\right] - (y_n + k_1) = y_n + hk_2$$

modified Euler

(b) Computation of $f(x) = \int_0^x e^{-t^2}\,dt$ is equivalent to solving

(fund. theorem of calculus)

$$y'(t) = e^{-t^2}$$

$$y(0) = 0$$

So just apply modified Euler with $f(t,y) = e^{-t^2}$ and $h = \frac{1}{99}$

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5. By considering the scalar model initial value problem

$$\begin{aligned} y'(t) &= \lambda y(t), \quad t \in [0,T], \\ y(0) &= y_0, \end{aligned}$$

with real λ determine the linear stability domain (in this case a real interval) for the modified Euler method of Problem 4.

Use modified Euler with $f(t,y) = \lambda y(t)$

So $y_{n+1} = y_n + h k_2 = y_n + h f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2} k_1\right)$

$$= y_n + h\lambda\left(y_n + \frac{h}{2}\underbrace{\lambda y_n}_{=k_1}\right)$$

$$= y_n + h\lambda y_n + \frac{h^2\lambda^2}{2} y_n = \left(1 + h\lambda + \frac{(h\lambda)^2}{2}\right) y_n$$

Apply recursively, so that

$$y_n = \left(1 + h\lambda + \frac{(h\lambda)^2}{2}\right)^n y_0$$

For stability we need growth factor

$$\left| 1 + h\lambda + \frac{(h\lambda)^2}{2} \right| < 1$$

$$-1 < 1 + h\lambda + \frac{(h\lambda)^2}{2} < 1$$

$\Rightarrow h\lambda + \frac{(h\lambda)^2}{2} > -2$ and $h\lambda + \frac{(h\lambda)^2}{2} < 0$

$z = h\lambda$

$\Leftrightarrow z^2 + 2z + 4 > 0$ $\qquad \Leftrightarrow z(2+z) < 0$

$\Leftrightarrow (z+1)^2 + 3 > 0$ $\qquad \Leftrightarrow -2 < z < 0$

true for all z

Together: $-2 < h\lambda < 0$

or $\underline{0 < h < \frac{-2}{\lambda}}$ (since $\lambda < 0$)