Math 590 — Homework 2 — due Wednesday, Oct.22, 2014

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1. Find eigenvalues and eigenfunctions for the differential eigenvalue problem $\mathcal{L}\varphi(\boldsymbol{x}) = \mu\varphi(\boldsymbol{x})$ with $\mathcal{L} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with boundary conditions $\varphi(0) = \varphi(1) = 0$.

2. Consider $\mathcal{L}G(\boldsymbol{x}, \boldsymbol{z}) = \delta(\boldsymbol{x} - \boldsymbol{z})$ for the differential operator $\mathcal{L} = \frac{\mathrm{d}^4}{\mathrm{d}x^4}$ together with boundary conditions $G(0, z) = G(1, z) = G''(0, z) = G''(1, z) = 0$.

   (a) Show that the corresponding Green's kernel G is given by

   $$G(x, z) = \begin{cases} \frac{1}{6}x(1-z)\left(1 - x^2 - (1-z)^2\right), & 0 \le x \le z \le 1, \\ \frac{1}{6}z(1-x)\left(1 - z^2 - (1-x)^2\right), & 0 \le z \le x \le 1. \end{cases}$$

   (b) Verify that for any fixed z the kernel G is a cubic natural spline that interpolates zero at $x = 0$ and $x = 1$.

3. Fill in the details in the derivation of the closed form representation of the piecewise polynomial spline kernels K_β (see Chapter 6), i.e., show

   $$K_\beta(x, z) = (-1)^{\beta-1}\frac{2^{2\beta-1}}{(2\beta)!}\left[B_{2\beta}\left(\frac{|x-z|}{2}\right) - B_{2\beta}\left(\frac{x+z}{2}\right)\right], \quad 0 \le x, z \le 1.$$

   Make sure to also explain why this is a piecewise polynomial of degree $2\beta - 1$.

4. (a) Find the kernel $K_{\beta,\varepsilon,M}$ obtained by truncating the Mercer series of the iterated Brownian bridge kernel $K_{\beta,\varepsilon}$ at M terms such that $\|K_{\beta,\varepsilon} - K_{\beta,\varepsilon,M}\|_\infty < \epsilon$.

   (b) Build a table analogous to the one in Chapter 6 page 31 and compare the results.

   Hint: Consider the infinite series of the eigenvalues as a Riemman sum and use a corresponding integral to obtain a good upper bound.