2.5.1 a) $\nabla^2 u = 0$

$$\frac{\partial u}{\partial x}(0,y) = \frac{\partial u}{\partial x}(L,y) = u(x,0) = 0 \quad , u(x,H) = f(x)$$

Separation of variables: $u(x,y) = \varphi(x)h(y)$.

Then

(I) $\varphi''(x) = -\lambda\varphi(x) \quad , \quad \varphi'(0) = \varphi'(L) = 0$

(II) $h''(y) = \lambda h(y) \quad , \quad h(0) = 0$

(I) has eigenvalues and eigenfunctions

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad , \quad \varphi_n(x) = \cos\frac{n\pi x}{L} \quad , n = 1,2,3,\ldots$$

$$\lambda_0 = 0 \quad , \quad \varphi_0(x) = 1$$

Inserting λ_n into (II) gives

$\lambda_n = \left(\frac{n\pi}{2}\right)^2$: $\quad h_n''(y) = \underbrace{\left(\frac{n\pi}{L}\right)^2}_{>0} hy$

$$\Rightarrow h_n(y) = c_1\cosh\frac{n\pi y}{L} + c_2\sinh\frac{n\pi y}{L}$$

so that $h_n(0) = 0 = c_1$

and $h_n(y) = \sinh\frac{n\pi y}{2}$

$$\left.\begin{aligned} \lambda_0 = 0 : h_0(y) &= c_1 y + c_2 \\ h_0(0) &= 0 = c_2 \end{aligned}\right\} \Rightarrow h_0(y) = y$$

Superposition:

$$u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\sinh\frac{n\pi y}{L}$$

Use $u(x,H) = f(x)$ to get

$$A_0 = \frac{1}{HL}\int_0^L f(x)\,dx$$

$$A_n = \frac{2}{L\sinh\frac{n\pi H}{2}}\int_0^L f(x)\cos\frac{n\pi x}{2}\,dx \quad , n = 1,2,3\ldots$$

e) $u(0,y) = u(L,y) = 0$

$u(x,0) - \frac{\partial u}{\partial y}(x,0) = 0 \quad , \quad u(x,H) = f(x)$

Ansatz: $u(x,y) = \varphi(x)\, h(y)$, so that

(I) $\varphi''(x) = -\lambda \varphi(x)$, $\varphi(0) = \varphi(L) = 0$

(II) $h''(y) = \lambda h(y)$, $h(0) - h'(0) = 0$

Eigenfunctions + eigenvalues of (I):

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad \varphi_n(x) = \sin\frac{n\pi x}{L}, \quad n = 1,2,3,\ldots$$

With these (II) becomes $\quad h_n''(y) = \left(\frac{n\pi}{L}\right)^2 h_n(y)$

$$\Longrightarrow \quad h_n(y) = c_1 \cosh\frac{n\pi y}{L} + c_2 \sinh\frac{n\pi y}{L}$$

Also need $h_n'(y) = \frac{n\pi}{L}\left(c_1 \sinh\frac{n\pi y}{L} + c_2 \cosh\frac{n\pi y}{L}\right)$

Then $h_n(0) = h_n'(0) \iff c_1 = \frac{n\pi}{L} c_2$

and $\quad h_n(y) = \cosh\frac{n\pi y}{L} + \frac{L}{n\pi}\sinh\frac{n\pi y}{L}$

Superposition:

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\left(\cosh\frac{n\pi y}{L} + \frac{L}{n\pi}\sinh\frac{n\pi y}{L}\right)$$

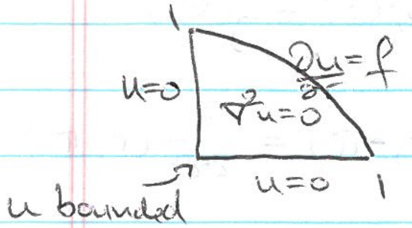
Use $u(x,H) = f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\left(\cosh\frac{n\pi H}{y} + \frac{L}{n\pi}\sinh\frac{n\pi H}{L}\right)$

to get

$$B_n = \frac{2}{L\left(\cosh\frac{n\pi H}{L} + \frac{L}{n\pi}\sinh\frac{n\pi H}{L}\right)} \int_0^L f(x) \sin\frac{n\pi x}{L}\, dx$$

$n = 1,2,3,\ldots$

2.5.5 c) $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$



The Ansatz $u(r,\theta) = R(r)\Theta(\theta)$ leads to

(I) $r^2 R'' + r R' - \lambda R = 0$, $|R(0)| < \infty$

(II) $\Theta'' = -\lambda\Theta$, $\Theta(0) = \Theta(\tfrac{\pi}{2}) = 0$

(II) is an eigenvalue problem with $(L = \tfrac{\pi}{2})$

$$\lambda_n = \left(\frac{n\pi}{\pi/2}\right)^2 = (2n)^2, \quad \Theta_n(\theta) = \sin 2n\theta, \quad n = 1,2,3\ldots$$

Inserting the eigenvalues in (I) we get

$$r^2 R'' + r R' - (2n)^2 R = 0$$

This Cauchy-Euler equation has solutions

$$R(r) = c_1 r^{2n} + c_2 r^{-2n}$$

$|R(0)| < \infty$ (boundedness at origin) $\Longrightarrow c_2 = 0$

Superposition: $u(r,\theta) = \sum_{n=1}^{\infty} B_n r^{2n} \sin 2n\theta$

Also need $\frac{\partial u}{\partial r}(r,\theta) = \sum_{n=1}^{\infty} 2n B_n r^{2n-1} \sin 2n\theta$

Then $\frac{\partial u}{\partial r}(1,\theta) = f(\theta) = \sum_{n=1}^{\infty} 2n B_n \sin 2n\theta$

and
$$B_n = \frac{2}{2n\,\pi/2}\int_0^{\pi/2} f(\theta)\sin 2n\theta\, d\theta$$
$$= \frac{2}{n\pi}\int_0^{\pi/2} f(\theta)\sin 2n\theta\, d\theta$$

2.5.8 a)

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$u(a,\theta) = f(\theta), \quad u(b,\theta) = g(\theta)$$

Also: perfect thermal contact

$$u(r,-\pi) = u(r,\pi), \quad \frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi)$$

Since we have 2 nonhomogeneous conditions this problem is similar to the problem discussed in class with the 4 nonhomogeneous conditions on the rectangle.
We need to solve two subproblems and then use superposition.

$u_1(a,\theta) = f(\theta)$ and $u_2(a,\theta) = 0$

$u_1(b,\theta) = 0$ $\qquad u_2(b,\theta) = g(\theta)$

First u_1:

$$\Theta'' = -\lambda\Theta, \quad \Theta(-\pi) = \Theta(\pi), \quad \Theta'(-\pi) = \Theta'(\pi)$$

$$\Rightarrow \lambda_n = n^2, \quad \Theta_n(\theta) = c_1\cos n\theta + c_2 \sin n\theta$$
$$\lambda_0 = 0, \quad \Theta_0(\theta) = 1.$$

and $r^2 R_n'' + rR_n' - n^2 R_n = 0$ with BC $R_n(b) = 0$

$n=0$: $R_0(r) = c_1 + c_2 \ln r$

BC: $R_0(b) = c_1 + c_2 \ln b = 0 \Rightarrow c_1 = -c_2 \ln b$

and so

$$R_0(r) = c_1(\ln r - \ln b) = c_1 \ln\frac{r}{b}$$

$\underline{n>0}$: $R_n(r) = c_3 r^n + c_4 r^{-n}$

$$R_n(b) = 0 = c_3 b^n + c_4 b^{-n} \Rightarrow c_3 = -c_4 b^{-n-n} = c_4 b^{-2n}$$

Therefore $R_n(r) = c_4\left(b^{-2n} r^n - r^{-n}\right)$

$$= b^{-n} c_4 \left(\left(\frac{r}{b}\right)^n - \left(\frac{b}{r}\right)^n\right)$$

Superposition (for u_1) gives

$$u_1(r,\theta) = a_0 \ln\frac{r}{b} + \sum_{n=1}^{\infty}\left(a_n \cos n\theta + b_n \sin n\theta\right)\left[\left(\frac{r}{b}\right)^n - \left(\frac{b}{r}\right)^n\right]$$

Since $u_1(a,\theta) = f(\theta)$ we get

$$a_0 = \frac{1}{\ln\left(\frac{a}{b}\right) 2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

$$a_n = \frac{1}{\pi\left[\left(\frac{a}{b}\right)^n - \left(\frac{b}{a}\right)^n\right]}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta$$

$$b_n = \frac{1}{\pi\left[\left(\frac{a}{b}\right)^n - \left(\frac{b}{a}\right)^n\right]}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta$$

The work for u_2 is similar. You should get

$$R_0(r) = c_1 \ln\frac{r}{a}, \quad R_n(r) = a^{-n} c_4\left(\left(\frac{r}{a}\right)^n - \left(\frac{a}{r}\right)^n\right)$$

So that superposition (with the same Θ_n) gives

$$u_2(r,\theta) = \tilde{a}_0 \ln\frac{r}{a} + \sum_{n=1}^{\infty}\left(\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta\right)\left[\left(\frac{r}{a}\right)^n - \left(\frac{a}{r}\right)^n\right]$$

The Fourier coefficients $\tilde{a}_0$, $\tilde{a}_n$, $\tilde{b}_n$ are determined from $u_2(b,\theta) = g(\theta)$

2.5.10: Show the solution of
$\nabla^2 u = g$ in R
$u = f$ on ∂R
is unique.

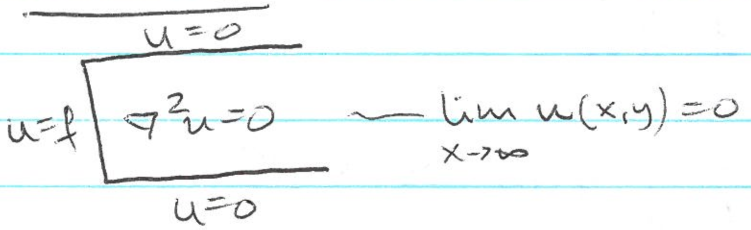
Take $w = u_1 - u_2$, where $\nabla^2 u_1 = g$, $\nabla^2 u_2 = g$ in R
$u_1 = f$, $u_2 = f$ on ∂R

Then $\nabla^2 w = \nabla^2 (u_1 - u_2)$
$= g - g = 0$ in R by linearity

and $w = u_1 - u_2 = f - f = 0$ on ∂R

The maximum principle then implies that $w = 0$ everywhere.

2.5.15 b) $\nabla^2 u = 0 \qquad 0 < x < \infty, \; 0 < y < H$



Conditions:
$u(x,0) = u(x,H) = 0$
$u(0,y) = f(y)$

Use Ansatz $u(x,y) = h(x)\varphi(y)$ so that

(I) $h''(x) = \lambda h(x)$, $\lim_{x\to\infty} h(x) = 0$

(II) $\varphi''(y) = -\lambda \varphi(y)$, $\varphi(0) = \varphi(H) = 0$

(II) has eigenvalues & eigenfunctions
$\lambda_n = \left(\frac{n\pi}{H}\right)^2$, $\varphi_n(y) = \sin\frac{n\pi y}{H}$, $n = 1, 2, 3 \ldots$

(I) has general solution
$h_n(x) = c_1 e^{\frac{n\pi x}{H}} + c_2 e^{-\frac{n\pi x}{H}}$

The decay condition $\lim_{x \to \infty} h_n(x) = 0$

implies $c_1 = 0$.

Superposition yields

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi y}{H} \, e^{-\frac{n\pi x}{H}}$$

with

$$b_n = \frac{2}{H} \int_0^H f(y) \sin \frac{n\pi y}{H} \, dy$$