RANDOMNESS: what is that and how to cope with it (with view towards financial markets)

Igor Cialenco

Dep of Applied Math, IIT

igor@math.iit.etu

MATH 100, Department of Applied Mathematics, IIT

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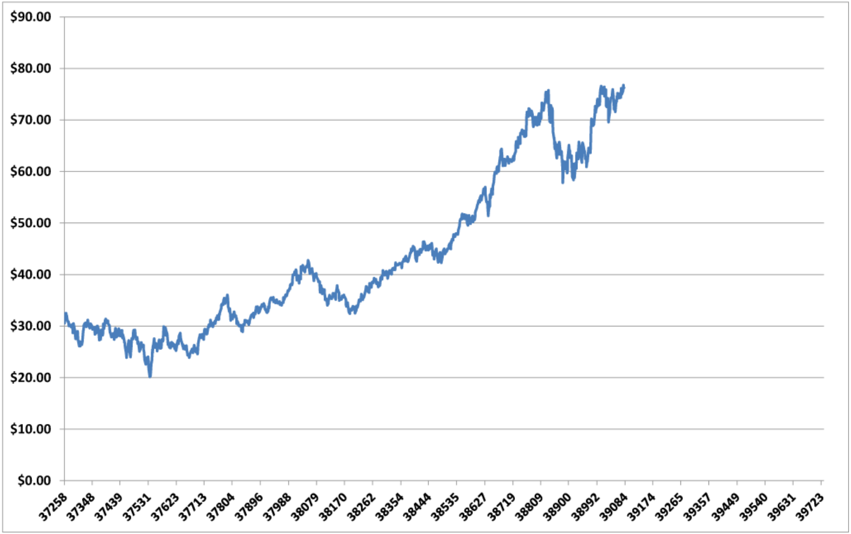
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Randomness is almost everywhere

Modeling it (the randomness) is FUN

What's randomness

- **Event(s) with Random Outcomes**

  Random, Stochastic, Uncertain, Chaotic, Unpredictable

- **Examples of Random Events:**

  flip a coin, temperature next Friday at noon, Dow Jones Industrial Average Tomorrow at 3:40pm, moving of a car in traffic, etc

- **Deterministic Outcomes:** - flipped coin, temp yesterday, number of days in a year 2089, etc

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"Probability, science originated in consideration of games of choice, should become the most important object of human knowledge"
Pierre Simon, Marquis de Laplace, 23 April 1749 - 5 March 1827, France

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  gambling - the rules are known, the sources of randomness are known

  stock market - the risk and randomness are changing, the rules and factors are unknown, we can only assume something about the randomness (the distribution of uncertainty)

- An attempt to describe various types of randomness "The Black Swan" by N.N.Taleb;

- David Aldous book review

  http://www.stat.berkeley.edu/~aldous/157/Books/taleb.html

- Andrew Gelman book review

  http://andrewgelman.com/2007/04/nassim_talebs_t/

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Both study the same objects and phenomena, but from very different points of view.

... an example will help to see the difference

Simplest example

Flip a coin

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Problem: you play a game in which you are paid \$5 if H and \$3 if T. How much should you pay to enter the game?
Answer: In a fair game you should pay the expected wining sum
$\mathbb{E}(\text{payoff}) = 5 \cdot p + 3 \cdot (1 - p)$

Flip a coin ... con't

- The model is done
- You can find about anything related to this model

  Flip the coin many times, look at the number of heads, number of consecutive heads, first time you have N heads and M tails, etc. All these probabilities can be evaluated

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- Some of the quantities of interest can be found by probabilistic methods (using in particular combinatorics) or by simulations
- You do not need a coin to simulate the game (computer can do)
- Computer Simulated Outcomes for flipping a coin
  $p = 0.7$ H H T H H H H T H H H H T T H H H
  $p = 0.1$ T T T H T T T T T T T T T T T T T T T T T T T H T T T T T T T T T H T T T T T T T T T T T T T H T T T T T T T H T T T T H T T
  $p = 0.5$ - fair coin H H T T T H H T H H T T H H T T T T H T T H T T H
- More on flipping a coin by Prof. Persi Diaconis

  http://news.stanford.edu/news/2004/june9/diaconis-69.html

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$$\widehat{p} = \frac{\text{\# of Heads}}{\text{\# of total observations}}$$

- More observation, better estimates (law of large numbers)

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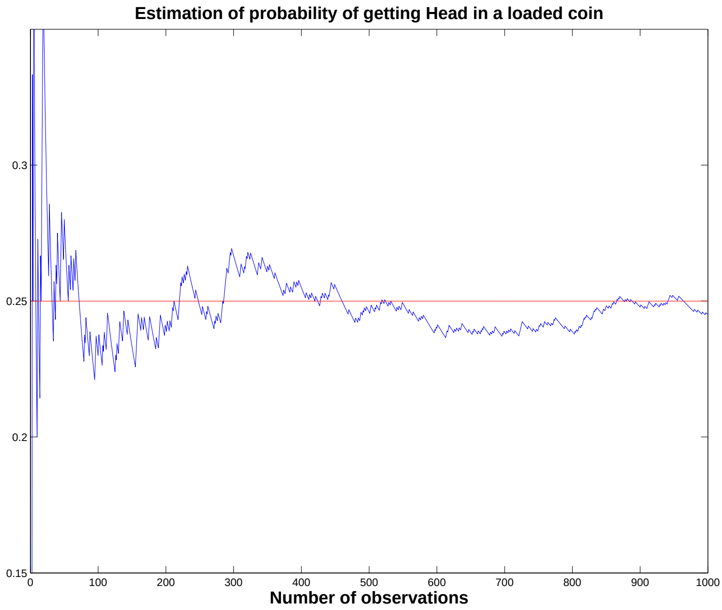
Statistics - based on past observations we try to find/inffer/estimate the probabilities of some events to happen. We try to make sense of past data.



Estimation of probability of getting Head in a loaded coin
0.3
0.25
0.2
0.15
0
100
200
300
400
500
600
700
800
900
1000
Number of observations

Same "game" ... same type of models ... same questions, same methods

- Roll a die and get paid the face value

  the model: six faces, six outcome $\Omega = \{1, 2, 3, 4, 5, 6\}$. Each face ends up with some probability $p_1, p_2, \ldots, p_6$. Note $p_1 + p_2 + \ldots + p_6 = 1$.

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- Fair value to enter the game? Expected payoff
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  Fair die, then $p_1 = p_2 = \ldots = p_6 = 1/6$ and $\mathbb{E}(\text{payoff}) = 3.5$
  Simulations 2 3 5 5 2 2 3 3 1 2 6 2 1 3 4 5 6 2 2 4 5 6 2 3 1

Other Casino type games. Same idea, as long as the rules are known.

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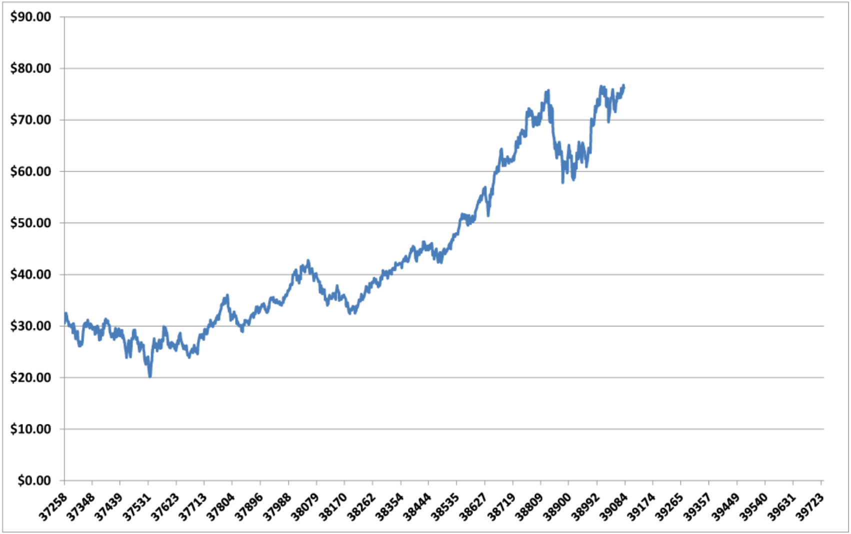
- Roulette? Easy, a fair die with 36 faces
- Blackjack? Also "easy", just more complicated combinatorics. No independency, so one can count the cards

Back to financial markets

predicting the stock price



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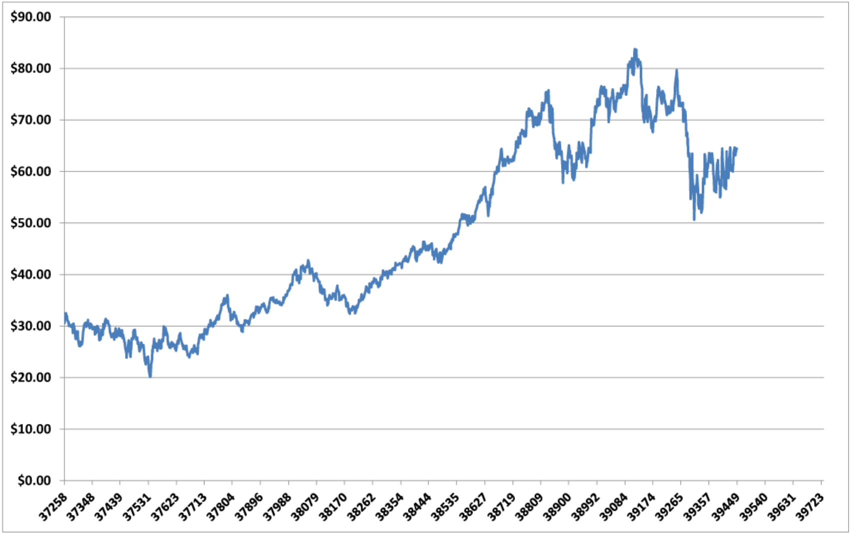

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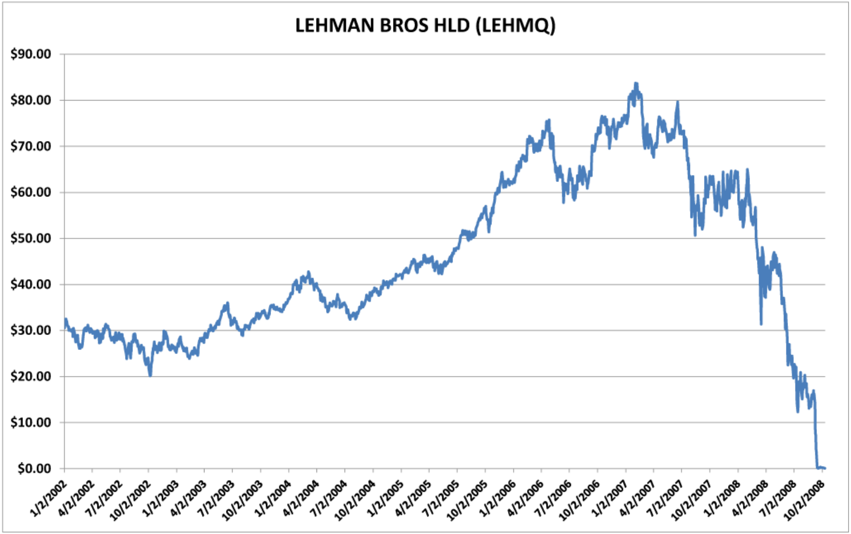
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What is so different in financial markets?

- The rules, sources of randomness, and sources of risk are changing.
- The factors driving the randomness in the market are unknown; we can only assume some properties about them (e.g. distribution).
- The stock price today already reflects all the past information. The price is based on **demand and supply**.
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HOWEVER!

still many things can be done

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No Arbitrage or No Free Lunch
(can not make money for sure out of nothing)

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Example (of arbitrage):
Bank ABC: deposit at 3.5% and borrow at 3.8% per year
Bank XYZ: deposit at 3% and borrow at 3.4% per year

Arbitrage: borrow, say $10,000 from XYZ, and deposit into ABC. This costs $0 at initiation. Close out the position at the end of the year, and get a sure profit of $10.

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Disclaimer: of course, we assumed that ABC and XYZ will not default within one year

Hedging/Replication of derivative contract

- Bank PQR wants to buy today the following (future) contract: for no \$'s down today, to agree on a price of $\$K$, paid in one year, for getting one share of AAPL (Apple Inc) also in one year.

- Bank KLM wants to sell this contract. Assume that KLM has access to credit (can borrow) for 3.0% per year.

- **Question:** What is $\$K$ that KLM wants to charge PQR?

- **Answer:** The fair price $K = \$563.4718$.

Hedging/Replication of derivative contract

$$K = \text{`AAPL price today'} \times (1 + 0.03) = \$547.06 \times 1.03 = \$563.4718$$

- Why? Because KLM can replicate. Assume that KLM enters the contract.
  - Borrow $\$547.06$ for one year under 3%
  - Buy one share of AAPL
  - Zero cost today
  - In one year ....
  - Get $K = 563.4718$ from PQR in exchange for that share of AAPL
  - Return to the lender exactly $\$563.4718$ (which is initial borrowing of $\$547.06$ plus the interest of $\$16.4118$)

Simple complex case - modeling stock price

Idea: Stock price - a banking account, but random (why not?)
Banking account $B_t = B_0 \cdot e^{rt}$, with r - interest rate

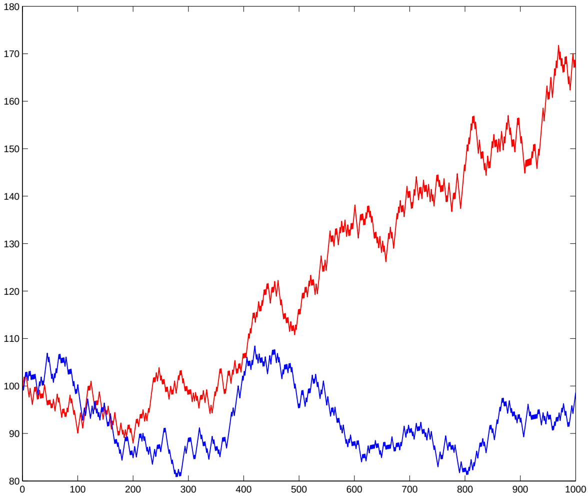
$$B_{t+\Delta t} = B_t e^{r\Delta t}$$

Stock - a random banking account, kind of ...

$$S_{t+\Delta t} = S_t e^{\mu\Delta t \pm \sigma\sqrt{\Delta t}}$$

with equal probabilities up or down ($\pm$).

Parameters μ, σ implied from the market or estimated historically.



Simulation of stock price using Black-Scholes-Merton model .

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How to model?

- Make simplifications
- Start from simple
- Keep track of general rules and laws of ‘nature’
- Use past data, but do not overuse it
- If no explicit solution, simulation usually helps

Thank You !

The end of the talk ...
but not of the story