Math 532 — Homework 2 — Due: Wednesday, January 28, 2015

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1. Assume that A and B are square matrices and that their product AB is invertible. Show that A and B must also be invertible.

2. Assume that A and B are invertible $n \times n$ matrices.

   (a) Show that
   $$\mathsf{A}^{-1} + \mathsf{B}^{-1} = \mathsf{A}^{-1}(\mathsf{A} + \mathsf{B})\mathsf{B}^{-1}.$$

   (b) What is the corresponding formula in the scalar case, i.e., take $n = 1$ and consider $\mathsf{A} = a$ and $\mathsf{B} = b$? Does the formula in (a) "make sense"?

   (c) Assume that $\mathsf{A} + \mathsf{B}$ is also invertible. What is $\left(\mathsf{A}^{-1} + \mathsf{B}^{-1}\right)^{-1}$?

3. True or false? Any $n \times n$ matrix can be expressed as the sum of two invertible matrices. Provide a proof if you think this is true, or a counterexample to show it is false.

4. (a) Compute the inverse of
   $$\mathsf{A} = \begin{pmatrix} 14 & 17 & 3 \\ 17 & 26 & 5 \\ 3 & 5 & 1 \end{pmatrix}.$$

   (b) Use the Sherman-Morrison formula and the result for A^{-1} from (a) to compute the inverse of
   $$\tilde{\mathsf{A}} = \begin{pmatrix} 14 & 17 & 3 \\ 17 & 24 & 5 \\ 3 & 5 & 1 \end{pmatrix}.$$

5. Consider the $(n+m) \times (n+m)$ block matrix $\mathsf{M} = \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{pmatrix}$, where A is $n \times n$ and invertible, D is $m \times m$, and B and C have appropriate sizes.

   Verify that
   $$\begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{pmatrix}^{-1} = \begin{pmatrix} \mathsf{A}^{-1} + \mathsf{A}^{-1}\mathsf{B}\mathsf{S}^{-1}\mathsf{C}\mathsf{A}^{-1} & -\mathsf{A}^{-1}\mathsf{B}\mathsf{S}^{-1} \\ -\mathsf{S}^{-1}\mathsf{C}\mathsf{A}^{-1} & \mathsf{S}^{-1} \end{pmatrix},$$
   where $\mathsf{S} = \mathsf{D} - \mathsf{C}\mathsf{A}^{-1}\mathsf{B}$ is the *Schur complement* of A in M.

6. Consider the linear system $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ with $n \times n$ nonsingular system matrix A and $n \times 1$ right-hand side vector $\boldsymbol{b}$.

   Now apply a rank-1 update of the form $\boldsymbol{c}\boldsymbol{d}^T$ with $n \times 1$ vectors $\boldsymbol{c}$ and $\boldsymbol{d}$ to A to get the matrix $\tilde{\mathsf{A}} = \mathsf{A} + \boldsymbol{c}\boldsymbol{d}^T$.

   Show that the solution of the linear system $\tilde{\mathsf{A}}\tilde{\boldsymbol{x}} = \boldsymbol{b}$ is given by
   $$\tilde{\boldsymbol{x}} = \boldsymbol{x} - \frac{\boldsymbol{y}\boldsymbol{d}^T\boldsymbol{x}}{1 + \boldsymbol{d}^T\boldsymbol{y}},$$
   where $\boldsymbol{y}$ is the solution of $\mathsf{A}\boldsymbol{y} = \boldsymbol{c}$.

7. *Smoothing splines* are a popular tool for statistical data analysis and prediction. They can be expressed in the form

$$s(\boldsymbol{x}) = \boldsymbol{k}(\boldsymbol{x})^T \boldsymbol{c}, \tag{1}$$

where $\boldsymbol{k}(\boldsymbol{x})^T = \begin{pmatrix} K_1(\boldsymbol{x}) & \cdots & K_n(\boldsymbol{x}) \end{pmatrix}$ is a vector of values of basis functions (or *kernels*) used to represent the prediction model. The unknown expansion coefficients $\boldsymbol{c}$ are obtained by solving a linear system of the form

$$(\mathsf{K} + \mu \mathsf{I})\, \boldsymbol{c} = \boldsymbol{y},$$

where K is a so-called *kernel matrix* whose entries are $[\mathsf{K}]_{ij} = K_j(\boldsymbol{x}_i)$, $i, j = 1, \ldots, n$, i.e., the kernels evaluated at the locations $\boldsymbol{x}_i$ at which the data is sampled. Furthermore, $\boldsymbol{y} = \begin{pmatrix} y(\boldsymbol{x}_1) & \cdots & y(\boldsymbol{x}_n) \end{pmatrix}^T$ represents the given data values, I is an $n \times n$ identity matrix, and μ is a (fixed) *smoothing parameter*.

An alternative approach to data analysis and prediction is the so-called *kriging* (or radial basis function) interpolation approach, which is also of the form (1). However, its expansion coefficients $\boldsymbol{c}$ are obtained by solving a linear system of the form

$$\mathsf{K}\boldsymbol{c} = \boldsymbol{y},$$

where $\boldsymbol{y}$ and K are as above.

Use the results on the inverse of sums of matrices to show that — for given data $\boldsymbol{y}$, kernel matrix K, and smothing parameter μ — the smoothing spline fit of $\boldsymbol{y}$ can also be interpreted as the kriging fit of appropriately smoothed data $\tilde{\boldsymbol{y}}$. What is the relation of $\tilde{\boldsymbol{y}}$ to $\boldsymbol{y}$?