Math 532 — Homework 4 — Due: Wednesday, February 11, 2015

1. Let
$$\mathsf{A} = \begin{pmatrix} 1 & 1 & 5 \\ 2 & 0 & 6 \\ 1 & 2 & 7 \end{pmatrix}, \qquad \mathsf{B} = \begin{pmatrix} 1 & -4 & 4 \\ 4 & -8 & 6 \\ 0 & -4 & 5 \end{pmatrix}.$$
Determine the four fundamental subspaces for A and B. Are there any of these subspaces that the two matrices share?

2. Let $\boldsymbol{p}$ be one particular solution of a linear system $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$.

   (a) Explain the significance of the set
   $$\boldsymbol{p} + N(\mathsf{A}) = \{\boldsymbol{p} + \boldsymbol{h} : \ \boldsymbol{h} \in N(\mathsf{A})\}.$$

   (b) Sketch a picture of $\boldsymbol{p} + N(\mathsf{A})$ in $\mathbb{R}^3$ when (i) $\operatorname{rank}(\mathsf{A}_{3\times 3}) = 1$, (ii) $\operatorname{rank}(\mathsf{A}_{3\times 3}) = 2$.

3. Suppose you are conducting a study of the behavior of the customers at your hot new online bookstore specializing in math books. To this end you record the number of monthly purchases of each of your customers in a matrix P, where each entry $[\mathsf{P}]_{ij}$ denotes the number of purchases made by customer $\#i$ in month $\#j$.

   Assume that you have a total of 500 customers and that you survey them for six months. What is the maximal number of "typical customers" in the sense that the purchasing habits of everyone else must be a combination of the habits of just those "typical" ones?

4. To elaborate on the connection between diagonally dominant matrices and LU factorization, show that no pivoting is required when computing the LU factorization of a diagonally *column-dominant* matrix A (i.e., A^T is diagonally dominant).

   Do this by considering A in block form
   $$\mathsf{A} = \begin{pmatrix} \alpha & \boldsymbol{b}^T \\ \boldsymbol{c} & \mathsf{D} \end{pmatrix},$$
   perform one step of Gaussian elimination (i.e., zero out $\boldsymbol{c}$), and then show that the resulting Schur complement (recall HW#2) is diagonally column dominant provided that A was so too. An inductive argument then completes the proof.

5. Explain what is going on with the MATLAB output

```
>> rank(vander(linspace(0,1,20)))
ans =
    19
>> rank(vander(linspace(0,1,25)))
ans =
    21
>> rank(vander(linspace(0,1,30)))
ans =
    22
```

Here `x=linspace(a,b,n)` creates a vector of n evenly spaced numbers in the interval $[a, b]$ and `vander(x)` creates the $n \times n$ Vandermonde matrix based on the points $x_1, \ldots, x_n \in \mathbb{R}$ contained in the vector `x`. What answers would you expect based on our discussion in class?