MATH 100 – Introduction to the Profession

Linear Equations in MATLAB

Greg Fasshauer

Department of Applied Mathematics
Illinois Institute of Technology

Fall 2012

Where do systems of linear equations come up?

Where do systems of linear equations come up?

Everywhere!

Where do systems of linear equations come up?

Everywhere!

- They appear straightforwardly in
  - analytic geometry (intersection of lines and planes),
  - traffic flow networks,
  - Google page ranks,
  - linear optimization problems,
  - statistical data fitting,
  - Leontief's input-output model in economics,
  - electric circuit problems,
  - the steady-state analysis of a system of chemical or biological reactors,
  - the structural analysis of trusses (see Exercise 5.6),
  - and many other applications.

Where do systems of linear equations come up?

Everywhere!

- They appear straightforwardly in
  - analytic geometry (intersection of lines and planes),
  - traffic flow networks,
  - Google page ranks,
  - linear optimization problems,
  - statistical data fitting,
  - Leontief's input-output model in economics,
  - electric circuit problems,
  - the steady-state analysis of a system of chemical or biological reactors,
  - the structural analysis of trusses (see Exercise 5.6),
  - and many other applications.
- They are solved as the main step in the linearization of many nonlinear systems (in the solution of differential equations, optimization, etc.).

Where do systems of linear equations come up?

Everywhere!

- They appear straightforwardly in
  - analytic geometry (intersection of lines and planes),
  - traffic flow networks,
  - Google page ranks,
  - linear optimization problems,
  - statistical data fitting,
  - Leontief's input-output model in economics,
  - electric circuit problems,
  - the steady-state analysis of a system of chemical or biological reactors,
  - the structural analysis of trusses (see Exercise 5.6),
  - and many other applications.
- They are solved as the main step in the linearization of many nonlinear systems (in the solution of differential equations, optimization, etc.).
- One can certainly refer to them as one of the workhorses of applied mathematics.

Representation of Linear Systems

- Equation form:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + x_2 + 4x_3 &= 1 \\ 3x_1 + 4x_2 + x_3 &= 1 \end{aligned}$$

Representation of Linear Systems

- Equation form:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + x_2 + 4x_3 &= 1 \\ 3x_1 + 4x_2 + x_3 &= 1 \end{aligned}$$

- Matrix form: $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$, with

$$\mathsf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Representation of Linear Systems

- Equation form:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + x_2 + 4x_3 &= 1 \\ 3x_1 + 4x_2 + x_3 &= 1 \end{aligned}$$

- Matrix form: $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$, with

$$\mathsf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Remark

We always think of vectors as column vectors. If we need to refer to a row vector we use the notation $\boldsymbol{x}^T$ (in mathematics) or `x'` (in MATLAB).

Never use A^{-1} to solve $A\boldsymbol{x} = \boldsymbol{b}$

Never use A^{-1} to solve $A\boldsymbol{x} = \boldsymbol{b}$

In linear algebra (MATH 332) you will learn that the solution of

$$A\boldsymbol{x} = \boldsymbol{b}$$

is given by

$$\boldsymbol{x} = A^{-1}\boldsymbol{b}.$$

Never use A^{-1} to solve $A\boldsymbol{x} = \boldsymbol{b}$

In linear algebra (MATH 332) you will learn that the solution of

$$A\boldsymbol{x} = \boldsymbol{b}$$

is given by

$$\boldsymbol{x} = A^{-1}\boldsymbol{b}.$$

This is correct, but inefficient and more prone to roundoff errors.

Never use A^{-1} to solve $A\boldsymbol{x} = \boldsymbol{b}$

In linear algebra (MATH 332) you will learn that the solution of

$$A\boldsymbol{x} = \boldsymbol{b}$$

is given by

$$\boldsymbol{x} = A^{-1}\boldsymbol{b}.$$

This is correct, but inefficient and more prone to roundoff errors.

Always use special algorithms (preferably with some decomposition method such as LU, QR or SVD) to solve linear systems – even to compute the inverse itself (should you actually happen to need it).

Never use A^{-1} to solve $A\boldsymbol{x} = \boldsymbol{b}$ (cont.)

Example

Consider the trivial "system" $7x = 21$ and compare solution via the "inverse" and by straightforward division.

Never use A^{-1} to solve $A\boldsymbol{x} = \boldsymbol{b}$ (cont.)

Example

Consider the trivial "system" $7x = 21$ and compare solution via the "inverse" and by straightforward division.

Solution

- Division immediately yields $x = \frac{21}{7} = 3$.

Never use A^{-1} to solve $A\boldsymbol{x} = \boldsymbol{b}$ (cont.)

Example

Consider the trivial "system" $7x = 21$ and compare solution via the "inverse" and by straightforward division.

Solution

- Division immediately yields $x = \frac{21}{7} = 3$.
- Use of the "inverse" yields

$$x = 7^{-1} \times 21 = 0.142857 \times 21 = 2.999997.$$

Never use A^{-1} to solve $A\boldsymbol{x} = \boldsymbol{b}$ (cont.)

Example

Consider the trivial "system" $7x = 21$ and compare solution via the "inverse" and by straightforward division.

Solution

- Division immediately yields $x = \frac{21}{7} = 3$.
- Use of the "inverse" yields

$$x = 7^{-1} \times 21 = 0.142857 \times 21 = 2.999997.$$

Clearly, use of the inverse requires more work (first compute the inverse, then multiply it into the right-hand side), and it is less accurate. This holds even more so for larger systems of equations.

Never use A^{-1} to solve $A\boldsymbol{x} = \boldsymbol{b}$ (cont.)

Example

Consider the trivial "system" $7x = 21$ and compare solution via the "inverse" and by straightforward division.

Solution

- Division immediately yields $x = \frac{21}{7} = 3$.
- Use of the "inverse" yields

$$x = 7^{-1} \times 21 = 0.142857 \times 21 = 2.999997.$$

Clearly, use of the inverse requires more work (first compute the inverse, then multiply it into the right-hand side), and it is less accurate. This holds even more so for larger systems of equations.

Note that MATLAB is "smarter" than this, so that `7^(-1)*21` is still equal to 3.

How to solve linear systems by “division” in MATLAB

In order to mimic what we do (naturally) for a single equation, MATLAB provides two very sophisticated *matrix division* operators:

How to solve linear systems by “division” in MATLAB

In order to mimic what we do (naturally) for a single equation, MATLAB provides two very sophisticated *matrix division* operators:

- For systems $A\boldsymbol{x} = \boldsymbol{b}$, we have the backslash (or `mldivide`) operator, i.e.,

$$\boldsymbol{x} = A\backslash\boldsymbol{b},$$

How to solve linear systems by “division” in MATLAB

In order to mimic what we do (naturally) for a single equation, MATLAB provides two very sophisticated *matrix division* operators:

- For systems $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$, we have the backslash (or `mldivide`) operator, i.e.,

$$\boldsymbol{x} = \mathsf{A}\backslash\boldsymbol{b},$$

- and $\boldsymbol{x}^T\mathsf{A} = \boldsymbol{b}^T$ is solved using a forward slash or (`mrdivide`) operator, i.e.,

$$\boldsymbol{x}^T = \boldsymbol{b}^T/\mathsf{A}.$$

How to solve linear systems by "division" in MATLAB

In order to mimic what we do (naturally) for a single equation, MATLAB provides two very sophisticated *matrix division* operators:

- For systems $A\boldsymbol{x} = \boldsymbol{b}$, we have the backslash (or `mldivide`) operator, i.e.,

$$\boldsymbol{x} = A \backslash \boldsymbol{b},$$

- and $\boldsymbol{x}^T A = \boldsymbol{b}^T$ is solved using a forward slash or (`mrdivide`) operator, i.e.,

$$\boldsymbol{x}^T = \boldsymbol{b}^T / A.$$

Remark

These operators provide black boxes for the solution of (possibly even non-square or singular) systems of linear equations. They can be even extended to the cases $AX = B$ *(see Exercise 5.4) and* $XA = B$.

Cramer's rule is especially inefficient!

	Flops				
n	10^9 (Giga)	10^{10}	10^{11}	10^{12} (Tera)	10^{15} (Peta)
10	10^{-1}sec	10^{-2}sec	10^{-3}sec	10^{-4}sec	negligible
15	17 hours	1.74 hours	10.46 min	1 min	0.6 10^{-1} sec
20	4860 years	486 years	48.6 years	4.86 years	1.7 day
25	o.r.	o.r.	o.r.	o.r.	38365 years

Table : Computer times for solving $n \times n$ linear systems using Cramer's rule on various computers ("o.r." stands for "out of reach"). Borrowed from [Scientific Computing with MATLAB and Octave (2010)].

- "Flops" stands for "floating point operation per second".
- Standard desktop PCs and laptops (Intel i5, i7) currently can perform on the order of about 10-50 gigaflops.
- Today's fastest supercomputer (LLNL's IBM BlueGene/Q *Sequoia*, see `http://top500.org/`) runs at 16 petaflops.

The World's Simplest Impossible Problem [Cleve's Corner]

"I'm thinking of two numbers and their average is 3."

The World's Simplest Impossible Problem [Cleve's Corner]

"I'm thinking of two numbers and their average is 3."

What's the issue here?

The World's Simplest Impossible Problem [Cleve's Corner]

"I'm thinking of two numbers and their average is 3."

What's the issue here?

- Not enough information is given (it is under-determined).
- So the problem is not well-posed.
- It does not have a unique solution, but has infinitely many solutions.

The World's Simplest Impossible Problem [Cleve's Corner]

"I'm thinking of two numbers and their average is 3."

What's the issue here?

- Not enough information is given (it is under-determined).
- So the problem is not well-posed.
- It does not have a unique solution, but has infinitely many solutions.

In MATLAB we get different answers using different algorithms:

- Using the backslash operator:

```
A = [1/2 1/2],    b=3
x = A\b
```

The World's Simplest Impossible Problem [Cleve's Corner]

"I'm thinking of two numbers and their average is 3."

What's the issue here?

- Not enough information is given (it is under-determined).
- So the problem is not well-posed.
- It does not have a unique solution, but has infinitely many solutions.

In MATLAB we get different answers using different algorithms:

- Using the backslash operator:

```
A = [1/2 1/2],    b=3
x = A\b
```

- Using the pseudo-inverse:

```
A = [1/2 1/2],    b=3
x = pinv(A)*b
```

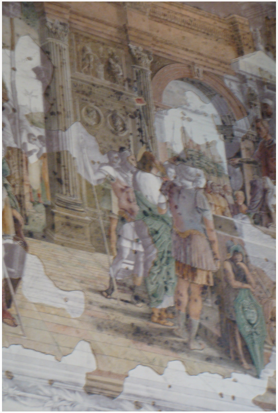
Compressed Sensing

An interesting recent article relating these two different algorithms (especially the backslash algorithm) to the hot research area of compressed sensing is `http://www.mathworks.com/company/newsletters/articles/clevescorner-compressed-sensing.html`.

There are additional links at the end of this article.

The main idea of compressed sensing is to be able to accurately reconstruct objects from very sparse information.

Massimo Fornasier is a researcher in Linz, Austria, who does lots of work in compressed sensing.



[Math enters the picture] describes the reconstruction (using matrix encoding and so-called *circular harmonics*) of the Italian renaissance frescoes by Andrea Mantegna in the Ovetari Chapel in Padua.

Summary scripts

The basic commands for dealing with systems of linear equations in MATLAB are summarized in

- `lin_sys.m` (solving linear systems on the MATH 100 website)
- `linear_recap.m` (on the ExM website)

References I

T. A. Driscoll.
Learning MATLAB.
SIAM, Philadelphia, 2009.
http://epubs.siam.org/ebooks/siam/other_titles_in_applied_mathematics/ot115

D. J. Higham and N. J. Higham.
MATLAB Guide (2nd ed.).
SIAM, Philadelphia, 2005.
http://epubs.siam.org/ebooks/siam/other_titles_in_applied_mathematics/ot92

C. Moler.
Numerical Computing with MATLAB.
SIAM, Philadelphia, 2004.
http://www.mathworks.com/moler/index_ncm.html

References II

C. Moler.
Experiments with MATLAB.
Free download at
http://www.mathworks.com/moler/exm/chapters.html

A. Quarteroni, F. Saleri and P. Gervasio.
Scientific Computing with MATLAB and Octave (3rd ed.).
Springer, Berlin, 2010.

M. Fornasier.
Mathematics enters the picture.
Proceedings of the conference *Mathknow 2008*. http://www.ricam.oeaw.ac.at/people/page/fornasier/mathsinpict.pdf.

C. Moler.
Cleve's Corner. The World's Simplest Impossible Problem.
The MathWorks News & Notes, Vol.4 No.2, 1990. http://www.mathworks.com/company/newsletters/news_notes/pdf/dec1990cleve.pdf

References III

The MathWorks.
MATLAB 7: Getting Started Guide.
http://www.mathworks.com/access/helpdesk/help/pdf_doc/matlab/getstart.pdf