MATH 100 – Introduction to the Profession

Proofs

Greg Fasshauer

Department of Applied Mathematics
Illinois Institute of Technology

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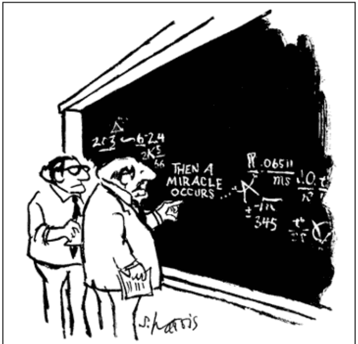
Outline[1]

[1] Most of this discussion is linked to [Devlin, Section 2.5] and [Gowers, Chapter 3].

"A *proof* *of a statement in mathematics is a* *logically sound argument that establishes the truth* *of the statement." [Devlin]*



"I think you should be more explicit here in step two."

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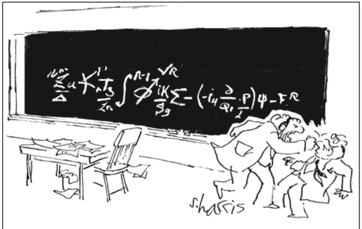

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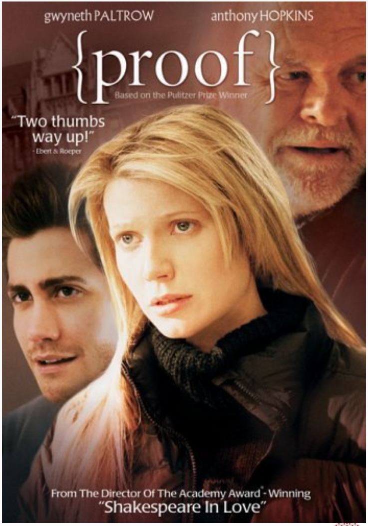

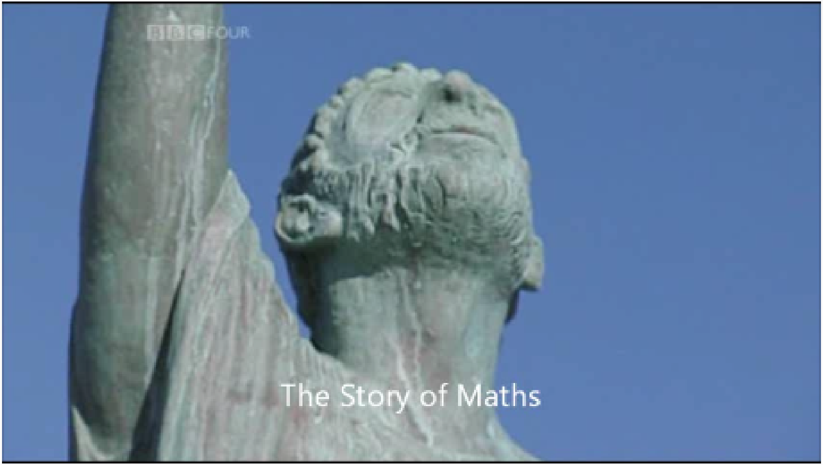
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The Story of Maths

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$$991n^2 + 1 \text{ is not a perfect square.}$$

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Therefore, this statement is *obviously* true.

Not so!

It takes a *looong* time to find a counter-example, but for

$$n = 12055735790331359447442538767$$

we have

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Conclusion

Simply checking (many) examples is not good enough to rigorously establish the truth of a statement. We need a mathematical proof.

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Proof.

To formalize this we assume m is the even integer and n is the odd one. Then the statement we want to prove is

$$(\forall m, n \in \mathbb{Z})\,[((m \text{ even}) \wedge (n \text{ odd})) \Rightarrow (mn \text{ even})]\,.$$

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$$mn = (2k)(2\ell + 1) = 2(2k\ell + k)$$

and since $2k\ell + k$ is an integer[a] we see that $mn = (2 \times \text{integer})$ is even. □

[a] It doesn't matter if even or odd.

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Any odd number can be represented as $n = 2k + 1$, for some integer k.
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Since $2k^2 + 2k$ is also an integer we have shown that n^2 is odd, and we are done. $\square$

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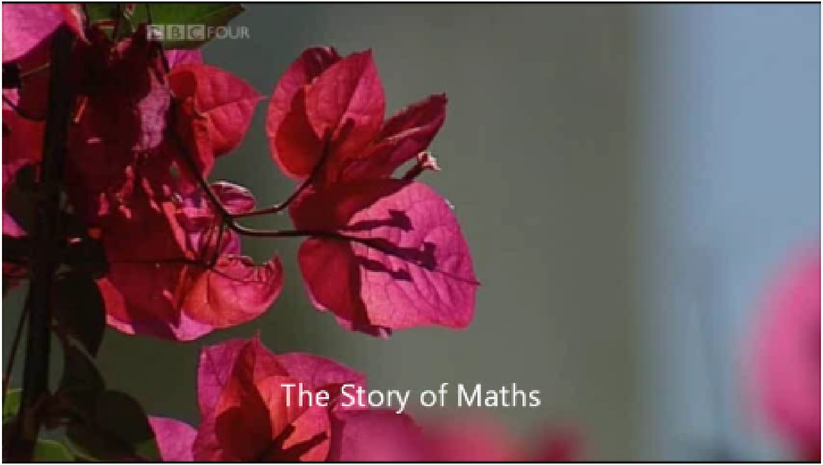
Some of the most famous examples of proofs by contradiction are:

- The proof that $\sqrt{2}$ is irrational (probably dating back to Aristotle ca. 350 B.C., see [Devlin, Section 2.5], [Gowers, Chapter 3]).
- The proof that there are infinitely many primes (dating back to Euclid ca. 300 B.C., see below).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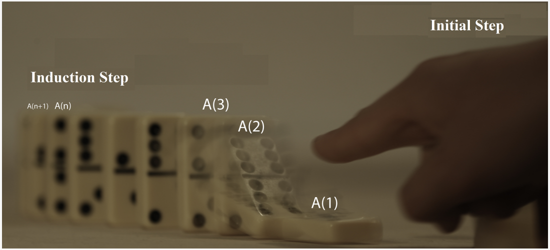
3. Combining (1) and (2) we conclude that the statement holds.



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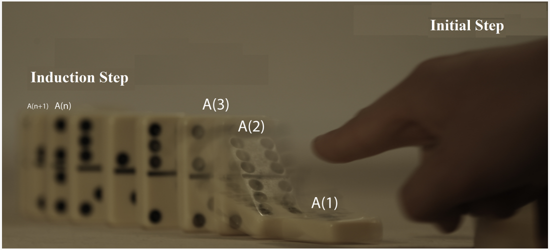

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This works because of the axioms that define the natural numbers.

Theorem (Exercise 2.5.7(a) in [Devlin], Gauss (9 years old))

For any natural number n, $1+2+3+\ldots+n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

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Since both sides of this equality evaluate to one we have ensured that the initial step holds.

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&= (n+1)\left(\frac{n}{2}+1\right)
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but this corresponds to the right-hand side of $A(n+1)$.
Since both the initial step and the induction step are true, the statement follows for all $n \in \mathbb{N}$.

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Little Gauss had to solve only the problem for $n = 100$:

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\begin{array}{ccccccccccccc}
1 & + & 2 & + & 3 & + & \ldots & + & 98 & + & 99 & + & 100 \\
100 & + & 99 & + & 98 & + & \ldots & + & 3 & + & 2 & + & 1 \\
\hline
\end{array}
$$

Gauss actually proved the above theorem directly (see [Gauss's Day of Reckoning]).
How would such a direct proof go?
Little Gauss had to solve only the problem for $n = 100$:

$$\begin{array}{ccccccccccccc} 1 & + & 2 & + & 3 & + & \ldots & + & 98 & + & 99 & + & 100 \\ 100 & + & 99 & + & 98 & + & \ldots & + & 3 & + & 2 & + & 1 \\ \hline 101 & + & 101 & + & 101 & + & \ldots & + & 101 & + & 101 & + & 101 \end{array}$$

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The number 101 is added 100 times, but we used two copies of the sum we wanted to compute, so

$$1 + 2 + 3 + \ldots + 98 + 99 + 100 = \frac{1}{2} 100 \cdot 101.$$

For general n the argument is analogous:

1	+	2	+	3	+	...	+	(n-2)	+	(n-1)	+	n
n	+	(n-1)	+	(n-2)	+	...	+	3	+	2	+	1
(n+1)	+	(n+1)	+	(n+1)	+	...	+	(n+1)	+	(n+1)	+	(n+1)

and we have

$$1 + 2 + 3 + \ldots + (n-2) + (n-1) + n = \frac{1}{2}n(n+1).$$

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This same problem can already be found (with a very similar solution) in [Problems to Sharpen the Young] by the English scholar Alcuin of York written in the 8$^{\text{th}}$ century.

Recall our problem from the beginning of the semester, where we conjectured the following:

Theorem

If the sequence $a_0, a_1, a_2, \ldots$ *satisfies*

$$a_{m+n} + a_{m-n} = \frac{1}{2}\left(a_{2m} + a_{2n}\right) \tag{*}$$

for all nonnegative integers m *and* n *with* $m \geq n$ *and* $a_1 = 1$*, then* $a_n = n^2$ *for all* $n \in \mathbb{N}_0$*.*

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for all nonnegative integers m and n with $m \geq n$ and $a_1 = 1$, then $a_n = n^2$ for all $n \in \mathbb{N}_0$.

While we computed a number of special values that might serve as the initial step of a mathematical induction proof for this problem, such as

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 4, \quad a_3 = 9, \quad \text{and even } a_{2m} = 4a_m,$$

ordinary induction does not suffice for this proof.

Instead we can use strong (or complete) induction. Here the induction step is:

- Assume that for an arbitrary n all of the following statements hold

$$A(1), A(2), \ldots, A(n)$$

and show that then $A(n+1)$ follows.

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Using the domino analogy, we're using not only the immediate predecessor to knock over the n^{th} domino, but we're allowed to use the combined force of all of its predecessors.

Proof (of sequence problem).

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We have (using m and $n = 1$ in $(*)$, and $a_{2m} = 4a_m$ and $a_2 = 4$)

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$$(a_{m+1} + a_{m-1} = 2a_m + 2) \Longleftrightarrow \left(a_{m+1} + (m-1)^2 = 2m^2 + 2\right)$$

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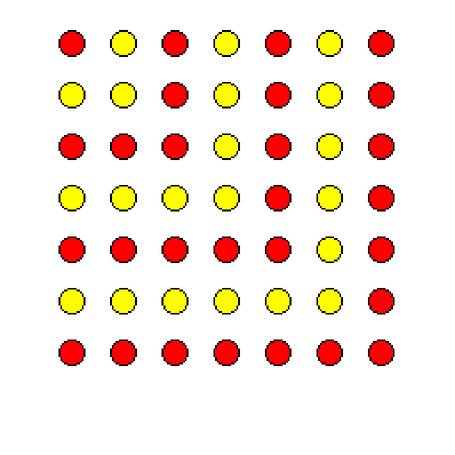
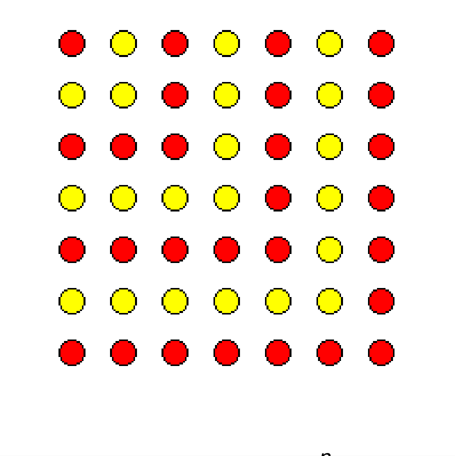
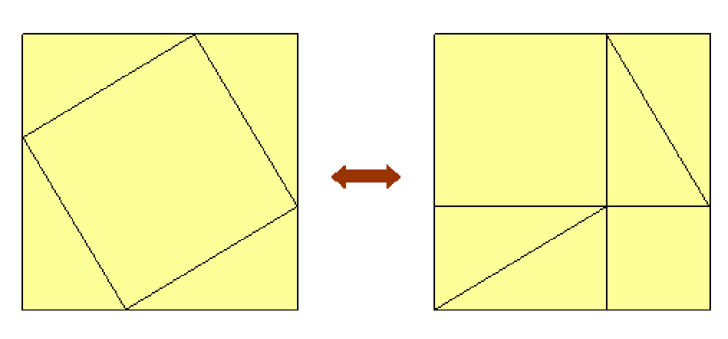
which corresponds to $A(m+1)$. $\square$

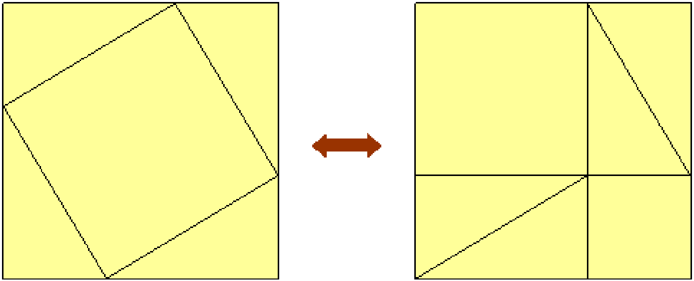




$$1 + 3 + 5 + \ldots + (2n - 1) = \sum_{k=1}^{n} (2k - 1) = n^2$$

See also HW problem 2.5.8(b) in [Devlin].





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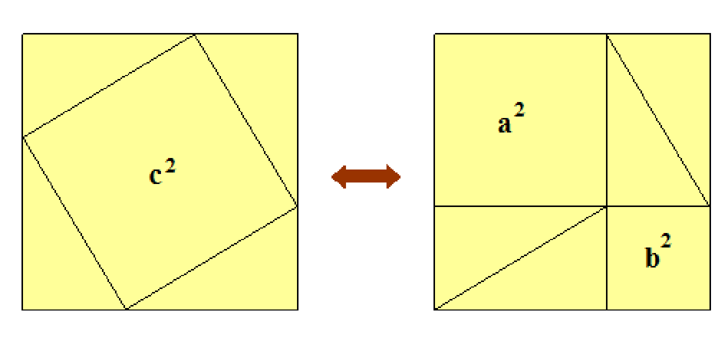

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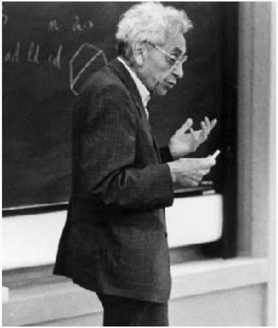
See also [Gowers, Chapter 3].


BBC FOUR
The Story of Maths

“This one’s from the book.” (Paul Erdős)



Refers to (famous) results with beautiful/elegant proofs.

Example

The Basel problem, first proved by Leonhard Euler in 1735:

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One way to prove this is via Fourier series (see MATH 461).

See [Proofs from THE BOOK] for three different proofs.

Theorem (Book IX, Prop. 20 of Euclid’s [Elements])

There are infinitely many primes.

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Euclid's Proof (a proof by contradiction).

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But p is not one of the p_i either since otherwise p would not only be a divisor of n, but also of the product $p_1 p_2 \cdots p_r$.

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Consequently, p would be a divisor of the difference $n - p_1 p_2 \cdots p_r = 1$.
But that is impossible, and so we have a contradiction, which means that set $\{p_1, \ldots, p_r\}$ cannot contain all primes. $\square$

The concept of proof is also relevant outside of mathematics.

In [The Elements of a Proposition] the authors analyze some of Abraham Lincoln’s speeches as they relate to Euclid’s [Elements].

Try this in MATLAB:

```
load penny.mat
contour(P,15)
colormap(copper)
axis ij square
```

Summary

You may see some of these proofs again in classes such as

- MATH 230 – Introduction to Discrete Math
- MATH 410 – Number Theory

Other classes that depend on lots of proofs are

- MATH 332 – Elementary Linear Algebra
- MATH 400 – Real Analysis
- MATH 420 – Geometry
- MATH 430/431 – Applied Algebra I/II
- MATH 453 – Combinatorics
- MATH 454 – Graph Theory

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